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CLOSED FORM, FINITE ELEMENT SOLUTIONS FOR PLATE VIBRATIONS

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SUMMARY

The convergence rates of eigenvalue solutions using two finite plate-bending elements are studied. The elements considered are the well-known 12-degree-of-freedom, non-conforming rectangular element and the 16-degree-of-freedom, conforming rectangular element. Three problems are analyzed: a square plate simply supported on two opposite sides, with the other two sides clamped, simply supported, or free. Closed form, finite element solutions for these problems are obtained by using shifting E-operators.

With few exceptions, eigenvalue solutions found with the non-conforming element converge from below the exact answers at an asymptotic rate of n^{-2}, where n is the number of elements on a side. However, since the array size needed for such convergence is very large, little can be said about the convergence rates for practical arrays. The conforming element solutions converge from above at a rate of n^{-4} for values of n larger than 6. A comparison of the errors involved in using these two elements shows that the conforming element is far superior to the non-conforming element.

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SYMBOLS

Symbol	Definition
a	length or width of square element
A,B,C,E,F,G,H,I,J	constants
D	plate stiffness constant, $= E\ h^3/12(1-\nu^2)$
E_r, E_s	shifting E-operators (see App. A)
h	plate thickness
i,j,k,l,m	constants
L	total length or width of plates considered, = na
n	number of elements on a side
p	number of half waves normal to simple supports
q	half the number of elements on a side
Q	compound shear boundary condition
r , s	incremental co-ordinate system for finite element arrays
t	time
x , y	continuous co-ordinate system of plates considered
X (x)	displacement function
W (x,y,t)	plate deflection
$\{W_1\}$, $\{W_2\}$	displacement vectors
w (x,y)	harmonic plate deflection
$\alpha, \beta, \xi, \phi, \delta, \sigma$	roots of characteristic equations
$\lambda_1 \ldots\ldots \lambda_{21}$	constants
$\epsilon_1 \ldots\ldots \epsilon_{25}$	constants
μ	mass per unit area of plate
ν	Poisson's ratio

SYMBOLS (Cont'd)

Symbol	Definition
ω	circular frequency
ω^2	non-dimensional frequency parameter, $= \mu\omega^2 L^4 / p^4 \pi^4 D$
γ^*	non-dimensional frequency parameter, $= \mu\omega^2 a^4 / 560\ D$
γ	non-dimensional frequency parameter, $= \gamma^*/2$
γ	non-dimensional frequency parameter, $= \mu\omega^2 L^4 /\ D$
[ ]	denotes a matrix
{ }	denotes a column vector

CLOSED FORM, FINITE ELEMENT SOLUTIONS FOR PLATE VIBRATIONS

1.0 INTRODUCTION

In recent years, a new, powerful approximate technique for solving complicated boundary value problems has evolved. This is the so-called finite element method, and it has been extensively developed for the analysis of structures. In this finite element technique, the continuous system is replaced by a substitute system consisting of a number of finite elements linked together. Once the properties, stiffness, mass, etc. of the individual elements have been defined, the whole substitute system can be described by large matrix equations, readily soivable using modern computers.

The success of the finite element technique depends upon the behaviour of each element and on how few elements it takes to adequately model a real structure. It is important that the finite element solutions converge to the exact answers as the number of elements is increased, and that this convergence be rapid. It is also desirable to have some estimate of the accuracy of any solutions found using a specific array of elements.

For plate-bending problems, two of the most important rectangular elements developed are the 12-degree-of-freedom non-conforming element and the 16-degree-of-freedom conforming element. The first element, independently derived by several investigators, Lindberg[1)], Melosh[2)], Zienkiewicz and Cheung[3)], Dawe[4)], and Clough and Tocher[5)] amongst others, was one of the first successful plate-bending finite elements developed. This element is called non-conforming since two elements linked together will have continuous displacements along their common edge, but will not have continuous normal slopes. More recently, a 16-degree-of-freedom element has been independently derived by Bogner, Fox and Schmit[6)], Butlin and Leckie[7)], and later by Mason[8)]. This element has the added property that two elements linked together have continuous normal slopes along their common edge as well as continuous displacements.

Recent literature has clarified to some extent the criteria that assure convergence of the finite element approximation to the true solution as the array is refined. A sufficient condition is that the element be capable of representing a constant strain condition; in the case of plate bending this should be a constant curvature condition, pure bending, or pure twist. This includes a condition of zero strain that corresponds to rigid body displacements. A further condition is that the element be conforming. This condition assures monotonic convergence of the total potential energy (Ref. 9,10,11). Both elements considered in this study satisfy the first criterion and, hence, yield results that must converge to correct solutions. The 16-degree-of-freedom element also satisfies the second criterion and, hence, must provide monotonic convergence of potential energy for static problems. Indeed, it is possible, following the arguments presented by Cowper et al.[11)], to show that the rate of convergence should be proportional to n^{-4}, where n is the number of elements on the side of an array. However, at this time, an extension of this general convergence proof to cover dynamic problems is still lacking.

The convergence of the 12-degree-of-freedom model has been studied by Walz, Fulton and Cyrus[12)] who show that for simply-supported square plates, both the static and dynamic convergence rates should be n^{-2}. For large values of n, they also provide an estimate of the errors involved in using these elements. This convergence study is somewhat limited, since only one boundary condition is considered, and the results hold true only for large arrays.

A more complete study of the dynamic convergence rates of these two elements is given in this report. Three problems are studied; a square plate simply supported on two opposite sides, with the other two sides simply supported, clamped, and free. The problems chosen all have exact solutions, so that the finite element errors may be systematically studied. A closed form type of solution using finite shifting E-operators is developed, so that solutions for large arrays can be found with little computational effort. This type of solution also reveals results of general interest that might easily be masked by a mass of arithmetical detail. Twenty eigenvalues are found for each of the problems, using each of the elements. The number of elements on a side is varied from 2 to 20, and the effects of reducing boundary conditions on some edges are investigated.

2.0 EXACT SOLUTIONS

The method for finding exact solutions to these three problems will now be given. The well-known differential equation governing the free vibrations of plates is (Timoshenko[13], p. 334)

$$\nabla^4 W = - \frac{\mu}{D}\frac{\partial^2 W}{\partial t^2} \tag{1}$$

where W(x,y,t) is the deflection of the plate, μ is the mass per unit area of the plate, D is the plate-bending rigidity, and t is the time. Assuming simple harmonic motion of the plate

$$W\ (x,\ y,\ t) = w\ (x,\ y) \sin \omega t \tag{2}$$

and substituting this into equation (1) leads to

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{\mu\omega^2}{D} w \tag{3}$$

Consider the plate shown in Figure 1, with the two opposite sides $y=0$ and $y=L$ simply supported. These boundary conditions are satisfied by a sine wave in the y-direction, so that

$$w\ (x,\ y) = X\ (x) \sin\ (p\pi y/L) \tag{4}$$

where p is the number of half waves in the y direction. Thus, the differential equation to be solved becomes

$$X'''' - 2\frac{p^2\pi^2}{L^2}X'' + \left(\frac{(p\pi)^4}{L^4} - \frac{\mu\omega^2}{D}\right) X = 0 \tag{5}$$

where $X'''' = \partial^4 X/\partial x^4$, etc.

Thus, the partial differential equation has been reduced to a solvable ordinary differential equation. Assuming that

$$X\ (x) = A \exp\ (\alpha\pi x/L) \tag{6}$$

and substituting this into equation (5) yields the characteristic equation

$$\alpha^4 - 2p^2\alpha^2 + p^4\ (1 - \bar{\omega}^2) = 0 \tag{7}$$

where $\bar{\omega}^2 = \mu\omega^2 L^4/p^4\pi^4 D$ is a non-dimensional frequency parameter. Solving for the roots of this quartic equation yields

$$\alpha^2 = p^2 \; (1 \pm \bar{\omega}) \tag{8}$$

If $\bar{\omega} > 1$, then

$$\alpha = \pm k \text{ , where } k^2 = p^2 \; (1 + \bar{\omega}) \tag{9}$$

and

$$\alpha = \pm im \text{ , where } m^2 = p^2 \; (\bar{\omega} - 1) \tag{10}$$

and i is the imaginary unit.

Using these four roots, the solution for X becomes

$$X(x) = A \cosh(k\pi x/L) + B \sinh(k\pi x/L) + C \cos(m\pi x/L) + E \sin(m\pi x/L) \tag{11}$$

If $\bar{\omega} < 1$, then

$$\alpha = \pm k \text{ , where } k^2 = p^2 \; (1 + \bar{\omega}) \tag{12}$$

and

$$\alpha = \pm \ell \text{, where } \ell^2 = p^2 \; (1 - \bar{\omega}) \tag{13}$$

so that

$$X(x) = F \cosh(k\pi x/L) + G \sinh(k\pi x/L) + H \cosh(\ell\pi x/L) + J \sinh(\ell\pi x/L) \tag{14}$$

The constants A, B, C, and E or F, G, H, and J can be determined by consideration of the boundary conditions along the two sides, $x = \pm L/2$. Two boundary conditions on each side are used to give four equations in the four unknown constants, and elimination of these gives a frequency equation for the particular case.

It is easier to consider symmetric and anti-symmetric solutions separately. For solutions that are symmetric in x about the y-axis, equation (4) becomes

$$w(x,y) = [A \cosh(k\pi x/L) + C \cos(m\pi x/L)] \sin(p\pi y/L) \tag{15}$$

while for the anti-symmetric case it becomes

$$w(x,y) = [B \sinh(k\pi x/L) + E \sin(m\pi x/L)] \sin(p\pi y/L) \tag{16}$$

Boundary conditions for the three types of edges considered are well known. For the clamped edge, the boundary conditions are

$$w(x, y)\Big|_{x = L/2} = 0$$

and (17)

$$\frac{\partial w}{\partial x}\Big|_{x = L/2} = 0$$

for the simply-supported edge, they are

$$w\,(x,\,y)\Big|_{x=L/2} = 0$$

and

$$M_x = \left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{x=L/2} = 0 \tag{18}$$

and for the free edge, they are

$$M_x = \left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{x=L/2} = 0$$

and (19)

$$Q = \left(V - \frac{\partial M_{xy}}{\partial y}\right) = \left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\,\frac{\partial^3 w}{\partial x \partial y^2}\right)\Bigg|_{x=L/2} = 0$$

The frequency equations for three types of boundary conditions are obtained by substituting equation (15) into the two boundary conditions, and then eliminating the constants from the ensuing equations. The transcendental equations are

$$k\,\tan(m\pi/2) + m\,\tanh(k\pi/2) = 0 \tag{20}$$

$$\cosh(k\pi/2)\,\cos(m\pi/2) = 0 \tag{21}$$

and

$$m\left(\frac{m^2 + (2-\nu)\,p^2}{m^2 + \nu p^2}\right)\tan(m\pi/2) + k\left(\frac{k^2 - (2-\nu)\,p^2}{k^2 - \nu p^2}\right)\tanh(k\pi/2) = 0 \tag{22}$$

for the clamped, simply-supported, and free boundaries, respectively.

Similar transcendental equations can be found for the anti-symmetric cases and for the cases where $\bar{\omega} < 1$.

The eigenvalues for the problems correspond to the roots or zero values of the transcendental equations, and may be found by an iterative process. A value of p, the number of half waves in the y-direction, is selected and values of the non-dimensional frequency parameter $\bar{\omega}$ are used to evaluate the transcendental equation being solved. When a zero crossing is found, an iterative procedure is used to obtain a precise value of frequency that makes the equation as close to zero as required. The exact non-dimensional frequencies found for these three problems are given later.

3.0 CLOSED FORM FINITE ELEMENT SOLUTIONS

The straightforward method of obtaining eigenvalues for various finite element gridworks is to set up the large system of simultaneous equations for a particular gridwork, apply boundary conditions, and solve the resulting eigenvalue problem on a digital computer. There are two difficulties in using such a procedure for the present study. Firstly, the size of the eigenvalue problem

rises rapidly with increase of the number of elements used, and hence the present study would involve a prohibitive amount of computation time. Secondly, results of general interest can be hidden in a mass of arithmetic detail.

A method of closed form solution utilizing shifting E-operators has therefore been adopted for this study. This method was developed by Lindberg[1] in a comparison study of several different finite elements. It was also used in a simplified form by Leckie[14]. The procedures are similar to those used in finding the exact solutions for the three problems. The equilibrium equations for an internal point of an assemblage of elements can be written using the stiffness and mass matrices of the adjacent elements. These equilibrium equations are expressed in terms of the generalized displacements of the surrounding element corners, and shifting E-operators are used to express these equations in terms of the generalized displacements of the single internal point. Expressions for these generalized displacements, sinusoidal in the direction normal to the simple supports and exponential in the other direction, are assumed and substituted into the equilibrium equations. A characteristic equation for the assumed displacement functions is then obtained, and general expressions for the displacement functions are found. These general functions can then be substituted into the generalized force equations for an edge point, thus obtaining both force and displacement equations for an edge. By selecting the appropriate boundary conditions, an approximate transcendental frequency equation for each of the three cases is found. The roots of these equations are found using iteration procedures.

These equations are a function of the number of elements used in the problem, and it is easy to vary this number and study the convergence of the approximate solutions. It is also possible to use a large number of elements without increasing the amount of computing time required for a solution.

3.1 Twelve-degree-of-freedom, Non-conforming Element

This element has three degrees of freedom per corner, ψ_x, ψ_y, and w/a, a total of 12 degrees of freedom. It is commonly called a non-conforming element, since two elements linked together have continuous displacements along their common edge, but do not have continuous normal slopes there. The element is well described in the literature (Ref. 1-5), so only the numerical values of the stiffness and mass matrices used in this study need be given. These stiffness and mass matrices for a square element of side a, and Poisson's ratio of ⅓, are given in Tables 1 and 2 respectively, where the displacement vector is

$$\{\psi_{x1},\ \psi_{y1},\ w_1/a,\ \psi_{x2}, \ldots\ldots\ldots\ldots\ldots\ldots w_4/a\}$$

The problem to be solved is shown in Figure 2 for the particular element assemblage of n = 2q = 6. Note that the co-ordinates r and s are not continuous, but rather are incremental and refer to assemblage points. Consider the internal point 5 of the assemblage shown in Figure 3. Since the displacements at the internal point are assumed to be continuous, it is possible to write a matrix equation for the forces at this point as:

$$\begin{Bmatrix} M_r \\ M_s \\ V\cdot a \end{Bmatrix}_5 = \frac{D}{45}\left[\begin{matrix} 17 & 0 & 39 & 44 & 0 & 0 & 17 & 0 & -39 & 56 & 0 & 192 & 272 & 0 & 0 \\ 0 & 17 & 39 & 0 & 56 & 192 & 0 & 17 & 39 & 0 & 44 & 0 & 0 & 272 & 0 \\ -39 & -39 & -66 & 0 & -192 & -408 & 39 & -39 & -66 & -192 & 0 & -408 & 0 & 0 & 1896 \end{matrix}\right.$$

$$\left.\begin{matrix} 56 & 0 & -192 & 17 & 0 & 39 & 44 & 0 & 0 & 17 & 0 & -39 \\ 0 & 44 & 0 & 0 & 17 & -39 & 0 & 56 & -192 & 0 & 17 & -39 \\ 192 & 0 & -408 & -39 & 39 & -66 & 0 & 192 & -408 & 39 & 39 & -66 \end{matrix}\right] \{W_i\}$$

$$\frac{\mu\omega^2 a^4}{25200}\left[\begin{array}{ccccccccccccccc} -30 & -28 & -116 & 80 & 0 & 0 & -30 & 28 & 116 & -120 & 0 & -548 & 320 & 0 & 0 \\ -28 & -30 & -116 & 0 & -120 & -548 & 28 & -30 & -116 & 0 & 80 & 0 & 0 & 320 & 0 \\ 116 & 116 & 394 & 0 & 548 & 2452 & -116 & 116 & 394 & 548 & 0 & 2452 & 0 & 0 & 13816 \end{array}\right.$$

$$\left.\begin{array}{cccccccccccc} -120 & 0 & 548 & -30 & 28 & -116 & 80 & 0 & 0 & -30 & -28 & 116 \\ 0 & 80 & 0 & 28 & -30 & 116 & 0 & -120 & 548 & -28 & -30 & 116 \\ -548 & 0 & 2452 & 116 & -116 & 394 & 0 & -548 & 2452 & -116 & -116 & 394 \end{array}\right]\{W_1\} \quad (23)$$

where

$$\{W_1\}^T = \{\psi_{r1}, \psi_{s1}, w_1/a, \psi_{r2}, \psi_{s2}, w_2/a, \ldots\ldots\ldots, \psi_{r9}, \psi_{s9}, w_9/a\}$$

is a 27-component vector of the displacements at assemblage points surrounding point 5.

Since only dynamic solutions with no external loads are required, then for equilibrium at any point these forces must equal zero.

A shifting E-operator (see App. A) is defined as

$$E_r^k\, E_s^m\, w_{r,s} = w_{r+k,\, s+m}$$

where $w_{r,s}$ is the displacement w at the point (r,s) in Figure 3. Then, using the r and s co-ordinates for points 1 to 9 shown in Figure 3, these equilibrium equations may be written as

$$\begin{aligned} &\{ (E_r^{-1} + E_r^1)(E_s^{-1} + E_s^1)(17 + 30\,\gamma^*) + E_r^0 (E_s^{-1} + E_s^1)(44 - 80\,\gamma^*) \\ &+ E_s^0 (E_r^{-1} + E_r^1)(56 + 120\,\gamma^*) + E_r^0 E_s^0 (272 - 320\,\gamma^*) \}\, \psi_{r5} \\ &+ \{ (E_r^{-1} - E_r^1)(E_s^{-1} - E_s^1)(28\,\gamma^*) \}\, \psi_{s5} \\ &+ \{ (E_r^{-1} + E_r^1)(E_s^{-1} + E_s^1)(39 + 116\,\gamma^*) + E_s^0 (E_r^{-1} - E_r^1)(192 + 548\,\gamma^*) \} w_5/a = 0 \end{aligned} \quad (24)$$

$$\begin{aligned} &\{ (E_r^{-1} - E_r^1)(E_s^{-1} - E_s^1)(28\,\gamma^*) \}\, \psi_{r5} \\ &+ \{ (E_r^{-1} + E_r^1)(E_s^{-1} + E_s^1)(17 + 30\,\gamma^*) + E_r^0 (E_s^{-1} + E_s^1)(56 + 120\,\gamma^*) \\ &+ E_s^0 (E_r^{-1} + E_r^1)(44 - 80\,\gamma^*) + E_r^0 E_s^0 (272 - 320\,\gamma^*) \}\, \psi_{s5} \\ &+ \{ (E_r^{-1} + E_r^1)(E_s^{-1} - E_s^1)(39 + 116\,\gamma^*) + E_r^0 (E_s^{-1} - E_s^1)(192 + 548\,\gamma^*) \}\, w_{5/a} = 0 \end{aligned} \quad (25)$$

$$\begin{aligned} &\{ (E_r^{-1} - E_r^1)(E_s^{-1} + E_s^1)(-39 - 116\,\gamma^*) - E_s^0 (E_r^{-1} - E_r^1)(192 + 548\,\gamma^*) \}\, \psi_{r5} \\ &+ \{ (E_r^{-1} + E_r^1)(E_s^{-1} - E_s^1)(-39 - 116\,\gamma^*) - E_r^0 (E_s^{-1} - E_s^1)(192 + 548\,\gamma^*) \}\, \psi_{s5} \\ &+ \{ (E_r^{-1} + E_r^1)(E_s^{-1} + E_s^1)(-66 - 394\,\gamma^*) - E_s^0 (E_r^{-1} + E_r^1)(408 + 2452\,\gamma^*) \\ &- E_r^0 (E_s^{-1} + E_s^1)(408 + 2452\,\gamma^*) + E_r^0 E_s^0 (1896 - 13816\,\gamma^*) \}\, w_{5/a} = 0 \end{aligned} \quad (26)$$

In these expressions, $\gamma^* = \mu\omega^2 a^4 / 560\, D$ is a non-dimensional frequency parameter.

Since the problems considered are all simply supported on two opposite sides, it is possible, as in the exact solution, to assume that the deflected shape in the s-direction is a sine wave. If it is assumed that the deflection in the r-direction varies exponentially, then

$$\begin{aligned}(w/a)_{r,s} &= A\, e^{\sigma r} \sin(p\pi s/n) \\ (\psi_r)_{r,s} &= B\, e^{\sigma r} \sin(p\pi s/n) \\ (\psi_s)_{r,s} &= C\, e^{\sigma r} \cos(p\pi s/n)\end{aligned} \tag{27}$$

where p is the number of half sine waves in the s-direction.

One of the valuable properties of E-operators is that they obey the rule

$$F_1(E_r)\, F_2\, (E_s)\, a^{kr}\, b^{ms} = a^{kr}\, b^{ms}\, F_1\, (a^k)\, F_2\, (b^m)$$

where F_1, F_2 are specified functions. Hence, it is easy to substitute exponential functions into E-operator equations. A table of E-operator transformations is given in Appendix A. Substituting the deflection equations (27) into the equilibrium equations (24-26) and using this transformation table yields the following three equations

$$\begin{aligned}&\sinh\sigma\,[-39\cos(p\pi/n) - 96 - \bar\gamma\,(58\cos(p\pi/n) + 137)\,]\quad A \\ &+ \{\cosh\sigma\,[17\cos(p\pi/n) + 28 + \bar\gamma\,(15\cos(p\pi/n) + 30)\,] \\ &+ 22\cos(p\pi/n) + 68 - \bar\gamma\,(20\cos(p\pi/n) + 40)\}\quad B \\ &- 14\,\bar\gamma\sinh\sigma\sin(p\pi/n)\quad C = 0\end{aligned} \tag{28}$$

$$\begin{aligned}&\sin(p\pi/n)\,\{\cosh\sigma\,(-39 - 58\bar\gamma) - 96 - 137\,\bar\gamma\,\}\,A + 14\,\bar\gamma\sinh\sigma\sin(p\pi/n)\quad B \\ &+ \{\cosh\sigma\,[17\cos(p\pi/n) + 22 + \bar\gamma\,(15\cos(p\pi/n) - 20)] + 28\cos(p\pi/n) + 68 \\ &+ \bar\gamma\,(30\cos(p\pi/n) - 40)\,\}\quad C = 0\end{aligned} \tag{29}$$

$$\begin{aligned}&\{\cosh\sigma\,[-66\cos(p\pi/n) - 204 - \bar\gamma\,(197\cos(p\pi/n) + 613)\,] \\ &- 204\cos(p\pi/n) + 474 - \bar\gamma\,(613\cos(p\pi/n) + 1727)\,\}\quad A \\ &+ \{\sinh\sigma\,[39\cos(p\pi/n) + 96 + \bar\gamma\,(58\cos(p\pi/n) + 137)]\,\}\quad B \\ &+ \sin(p\pi/n)\,\{\cosh\sigma\,(-39 - 58\,\bar\gamma\,) - 96 - 137\,\bar\gamma\,\}\quad C = 0\end{aligned} \tag{30}$$

where $\bar\gamma = \gamma^*/2$.

For a nontrivial solution for A, B, and C, the determinant of the coefficients must vanish, and this determinant simplifies to an equation cubic in cosh σ. This is the characteristic equation for these problems, and its three roots give three values for σ that can be used to determine w. For these problems, the roots are found to be

$$\cosh \sigma \leq 1$$

$$-1 \leq \cosh \sigma \leq 1$$

and

$$\cosh \sigma < -1$$

If $\cosh \sigma_1 > 1$, then $\sigma_1 = \pm \alpha$ say, and so

$$[(w/a)_{r,s}]_1 = (E \cosh \alpha r + F \sinh \alpha r) \sin(p\pi s/n) \tag{31}$$

If $-1 \leq \cosh \sigma_2 \leq 1$, then $\sigma_2 = \pm i\beta$ say, and so

$$[(w/a)_{r,s}]_2 = (G \cos \beta r + H \sin \beta r) \sin(p\pi s/n) \tag{32}$$

Lastly, consider $\cosh \sigma_3 < -1$. Say that $\cosh \sigma_3 = -\delta$, so that $\sigma_3 = \cosh^{-1}(\delta) + i\pi = \xi + i\pi$. Then

$$[(w/a)_{r,s}]_3 = (I(-1)^r \cosh \xi r + J(-1)^r \sinh \xi r) \sin(p\pi s/n) \tag{33}$$

Adding the three components gives a final expression for the deflection as

$$(w/a)_{r,s} = \{ E \cosh \alpha r + F \sinh \alpha r + G \cos \beta r + H \sin \beta r$$

$$+ I(-1)^r \cosh \xi r + J(-1)^r \sinh \xi r \} \sin(p\pi s/n) \tag{34}$$

It is interesting to note that the first four terms are similar to those found in the exact solution, while the last two terms arise as a result of the finite element approximation.

As in the exact solution, it is easiest to consider symmetric and anti-symmetric cases separately. If only symmetric solutions are considered, the deflection becomes

$$(w/a)_{r,s} = \{ E \cosh \alpha r + G \cos \beta r + I(-1)^r \cosh \xi r \} \sin(p\pi s/n) \tag{35}$$

By using the first two equilibrium equations, it is possible to solve for $(\psi_r)_{r,s}$ and $(\psi_s)_{r,s}$ in terms of $(w/a)_{r,s}$. If the above expression for the deflection is substituted into these expressions, and E-operator transformations are applied, then these expressions become

$$(\psi_r)_{r,s} = \{ E \lambda_1 \sinh \alpha r + G \lambda_2 \sin \beta r + I \lambda_3 (-1)^r \sinh \xi r \} \sin(p\pi s/n) \tag{36}$$

$$(\psi_s)_{r,s} = \{ E \lambda_4 \cosh \alpha r + G \lambda_5 \cos \beta r + I \lambda_6 (-1)^r \cosh \xi r \} \cos(p\pi s/n) \tag{37}$$

where the λ terms are complicated functions of α, β, ξ and $\frac{p\pi}{n}$. These terms are given in Appendix B.

Now the boundary conditions must be considered. The finite element assemblage for an edge point is shown in Figure 4. Using the stiffness and mass matrices of these elements, it is possible to write the equilibrium equations for this edge point as

$$\begin{Bmatrix} M_r \\ M_s \\ V\cdot a \end{Bmatrix}_{q,s} = \frac{D}{45}\left[\begin{array}{cccccccccccccccccc} 17 & 0 & 39 & 22 & 0 & -24 & 56 & 0 & 192 & 136 & 0 & -222 & 17 & 0 & 39 & 22 & 0 & -24 \\ 0 & 17 & 39 & 0 & 28 & 96 & 0 & 44 & 0 & 0 & 136 & 0 & 0 & 17 & -39 & 0 & 28 & -96 \\ -39 & -39 & -66 & -24 & -96 & -204 & -192 & 0 & -408 & -222 & 0 & 948 & -39 & 39 & -66 & -24 & 96 & -204 \end{array}\right]\{W_2\}$$

$$-\frac{\mu\,\omega^2 a^4}{25200}\left[\begin{array}{cccccccccccccccccc} -30 & -28 & -116 & 40 & -42 & -199 & -120 & 0 & -548 & 160 & 0 & -922 & -30 & 28 & -116 & 40 & 42 & -199 \\ -28 & -30 & -116 & 42 & -60 & -274 & 0 & 80 & 0 & 0 & 160 & 0 & 28 & -30 & 116 & -42 & -60 & 274 \\ 116 & 116 & 394 & -199 & 274 & 1226 & 548 & 0 & 2452 & -922 & 0 & 6908 & 116 & -116 & 394 & -199 & -274 & 1226 \end{array}\right]\{W_2\} \quad (38)$$

where

$$\{W_2\} = \{\psi_{r1}, \psi_{s1}, w_1/a, \psi_{r2}, \psi_{s2}, w_2/a, \ldots\ldots\ldots\ldots, \psi_{r6}, \psi_{s6}, w_6/a\}$$

Using E-operators, these equations for $(M_r)_{q,s}$, $(M_s)_{q,s}$, and $(V\cdot a)_{q,s}$ can be written in terms of $(\psi_r)_{q,s}$, $(\psi_s)_{q,s}$, and $(w/a)_{q,s}$. The expressions for these edge displacements are, since $r = q = n/2$,

$$(\psi_r)_{q,s} = \{E\,\lambda_1 \sinh \alpha q + G\,\lambda_2 \sin \beta q + I\,\lambda_3\,(-1)^q \sinh \xi q\} \sin(p\pi s/n)$$

$$(\psi_s)_{q,s} = \{E\lambda_4 \cosh \alpha q + G\,\lambda_5 \cos \beta q + I\,\lambda_6\,(-1)^q \cosh \xi q\} \cos(p\pi s/n) \quad (39)$$

$$(w/a)_{q,s} = \{E \cosh \alpha q + G \cos \beta q + I\,(-1)^q \cosh \xi q\} \sin(p\pi s/n)$$

Substituting these into the expressions for edge forces and using E-operator transformations gives

$$(M_r)_{q,s} = \frac{D}{45}\{\epsilon_1 E + \epsilon_2 G + \epsilon_3 I\} \sin(p\pi s/n)$$

$$(M_s)_{q,s} = \frac{D}{45}\{\epsilon_4 E + \epsilon_5 G + \epsilon_6 I\} \cos(p\pi s/n) \quad (40)$$

$$(V\cdot a)_{q,s} = \frac{D}{45}\{\epsilon_7 E + \epsilon_8 G + \epsilon_9 I\} \sin(p\pi s/n)$$

where the ϵ's are complicated functions of α, β and ξ, αq, βq, ξq, and $p\pi/n$. These functions are also given in Appendix B.

Now both the edge forces and the edge displacements have been defined (eq. (39) and (40)). By selecting the appropriate boundary conditions from these, transcendental equations can be found that yield the required eigenvalues of the three problems. These boundary conditions will now be considered.

Since there are three unknowns in the edge expressions, three boundary conditions must be satisfied. This is different than the exact solutions where only two boundary conditions could be satisfied, and arises because this type of solution is approximate. In the continuous case, if a function such as the displacement w is set equal to zero along an edge, then all the derivatives of w taken tangent to that edge are also automatically set equal to zero. This does not automatically follow in approximate solutions and, hence, it is reasonable to have to satisfy additional boundary conditions.

The boundary conditions for a clamped edge are

$$(w/a)_{q,s} = 0 \; ; \; (\psi_r)_{q,s} = (0 \; ; \; \psi_s)_{q,s} = 0 \tag{41}$$

Note that in the continuous case the third boundary condition $(\psi_s)_{q,s} = 0$ would automatically be satisfied by prescribing the first boundary condition, $(w/a)_{q,s}=0$, while in the approximate case this is not true.

For a simply-supported edge, the boundary conditions are

$$(w/a)_{q,s} = 0 \; ; \; (M_r)_{q,s} = 0 \; : \; (\psi_s)_{q,s} = 0 \tag{42}$$

Again, the first and third boundary conditions are intimately related. Finally, for a free edge, the boundary conditions are

$$(M_r)_{q,s} = 0 \; ; \; (M_s)_{q,s} = 0 \; ; \; (V \cdot a)_{q,s} = 0 \tag{43}$$

The difference between the boundary conditions for the continuous case and those for the approximate case is greatest for this last problem. In the continuous case, a compound shear condition was set equal to zero. This consisted of the shear force plus the rate of change of twisting moment. Here, both the shear force and the tangential bending moment are set equal to zero (the second and third conditions) but this tangential bending moment does not correspond to the twisting moment of the continuous theory.

For each problem, the appropriate edge expressions are set equal to zero, and the constants E, G, and I are eliminated to yield a transcendental frequency equation. As for the exact solutions, iteration procedures are used to find the roots of these expressions, and hence the eigenvalues. Note that each transcendental equation is a function of q, the number of finite elements on a half side of the problem. Thus, eigenvalues can be found for any desired value of q, and hence n, so it is possible to study the convergence of solutions as n is varied.

3.1.1 Two-root Approximate Solutions

In the exact solutions, only two boundary conditions on an edge may be satisfied. It has been shown that three conditions must be satisfied for the approximate solutions. An investigation was carried out to find the effect of leaving out the third boundary condition.

Consider the deflection expression, equation (39)

$$(w/a)_{r,s} = [E \cosh \alpha r + G \cos \beta r + I\,(-1)^r \cosh \xi r] \sin(p\pi s/n)$$

The first two terms are similar to those used in the exact solution, while the third term appears to be different. Assuming that it can be disregarded, the deflection becomes

$$(w/a)_{r,s} = [E \cosh \alpha r + G \cos \beta r] \sin(p\pi s/n) \qquad (44)$$

and slopes and edge forces are the same as before, with the third term dropped. The boundary conditions used in the two-root solutions are:

for the clamped edge

$$(w/a)_{q,s} = 0 \; ; \; (\psi_r)_{q,s} = 0 \qquad (45)$$

for the simply-supported edge

$$(w/a)_{q,s} = 0 \; ; \; (M_r)_{q,s} = 0 \qquad (46)$$

and for free edge

$$(M_r)_{q,s} = 0; \left\{ (V \cdot a)_{q,s} + \frac{(M_s)_{q,s+1} - (M_s)_{q,s-1}}{2a} \right\} = 0 \qquad (47)$$

Note that, in the free edge, the boundary condition used only approximates the classical boundary condition. As before, the application of the boundary conditions yields transcendental equations. The solutions of these two-root cases will be discussed later.

3.2 Sixteen-degree-of-freedom Conforming Element

This element, independently derived by Bogner, Fox and Schmit[6], Butlin and Leckie[7], and Mason[8], is called conforming since elements linked together have continuous normal slopes along their common edge as well as continuous displacements. There are four degrees of freedom at each corner point, ψ_x, ψ_y, $a\psi_{xy}$, and w/a. The stiffness and mass matrices for a square element of side a, and Poisson's ratio of ⅓ are given in Tables 3 and 4, where the displacement vector is

$$\{w_1/a, \psi_{x1}, \psi_{y1}, a\,\psi_{xy1}, w_2/a, \ldots\ldots\ldots\ldots, a\,\psi_{xy4}\}$$

Closed form solutions using this element are found, following the same procedure as before. Four equilibrium equations for an internal point are written, and by assuming displacement functions similar to equations (27), i.e.

$$(w/a)_{r,s} = A\, e^{\sigma r} \sin(p\pi s/n), \text{ etc.}$$

and applying E-operator transformations, a characteristic equation for the problems is found. This equation is quartic rather than cubic as before, and its four roots give values for σ that are used to determine (w/a). For these problems, it is found that all four roots are greater than -1. Typically

$$\cosh \sigma_1 \geq 1 \; ; \; \sigma_1 = \pm \alpha$$

$$\cosh \sigma_2 \geq 1 \; ; \; \sigma_2 = \pm \delta$$

$$\cosh \sigma_3 \geq 1 \; ; \; \sigma_3 = \pm \phi$$

$$-1 \leq \cosh \sigma_4 \leq 1 \; ; \; \sigma_4 = \pm i\beta$$

which yields the deflection expression

$$(w/a)_{r,s} = [A \cosh \alpha r + B \sinh \alpha r + C \cosh \delta r + D \sinh \delta r + E \cosh \phi r + F \sinh \phi r + G \cos \beta r + H \sin \beta r] \sin(p\pi s/n) \tag{48}$$

In some cases, two of the roots are between −1 and 1, and the deflection expression then contains four hyperbolic expressions and four trigonometric expressions.

As before, expressions for the slopes and twist are found in terms of the deflection using the equilibrium equations. The expression for (w/a) is substituted into these equations and, using E-operator transformations, they become

$$\begin{aligned}
(\psi_r)_{r,s} &= [A\, \lambda_{10} \sinh \alpha r + C\, \lambda_{11} \sinh \delta r + E\, \lambda_{12} \sinh \phi r + G\, \lambda_{13} \sin \beta r] \sin(p\pi s/n) \\
(\psi_s)_{r,s} &= [A\, \lambda_{14} \cosh \alpha r + C\, \lambda_{15} \cosh \delta r + E\, \lambda_{16} \cosh \phi r + G\, \lambda_{17} \cos \beta r] \cos(p\pi s/n) \\
(a\psi_{rs})_{r,s} &= [A\, \lambda_{18} \sinh \alpha r + C\, \lambda_{19} \sinh \delta r + E\, \lambda_{20} \sinh \phi r + G\, \lambda_{21} \sin \beta r] \cos(p\pi s/n)
\end{aligned} \tag{49}$$

where only symmetric terms are considered. No details of the equations or definition of the λ's are given, since the expressions involved are exceedingly cumbersome.*

Boundary conditions are then considered as before, and expressions for the edge forces are found as

$$\begin{aligned}
(M_r)_{q,s} &= D\, [\epsilon_{10} A + \epsilon_{11} C + \epsilon_{12} E + \epsilon_{13} G] \sin(p\pi s/n) \\
(M_s)_{q,s} &= D\, [\epsilon_{14} A + \epsilon_{15} C + \epsilon_{16} E + \epsilon_{17} G] \cos(p\pi s/n) \\
(M_{rs}/a)_{q,s} &= D\, [\epsilon_{18} A + \epsilon_{19} C + \epsilon_{20} E + \epsilon_{21} G] \cos(p\pi s/n) \\
(V \cdot a)_{q,s} &= D\, [\epsilon_{22} A + \epsilon_{23} C + \epsilon_{24} E + \epsilon_{25} G] \sin(p\pi s/n)
\end{aligned} \tag{50}$$

Again, no details are given.

It is apparent that there are four boundary conditions to satisfy, rather than three as before. These are:

for the clamped edge

$$(w/a)_{q,s} = 0\ ;\ (\psi_r)_{q,s} = 0\ ;\ (\psi_s)_{q,s} = 0\ ;\ (a\psi_{rs})_{q,s} = 0 \tag{51}$$

for the simply-supported edge

$$(w/a)_{q,s} = 0\ ;\ (\psi_s)_{q,s} = 0\ ;\ (M_r)_{q,s} = 0\ ;\ (M_{rs}/a)_{q,s} = 0 \tag{52}$$

---

*Full details will be supplied upon request.

and for the free edge

$$(M_r)_{q,s} = 0\ ;\ (M_s)_{q,s} = 0\ ;\ (M_{rs}/a)_{q,s} = 0\ ;\ (V \cdot a)_{q,s} = 0 \qquad (53)$$

For each of these cases, a 4 × 4 frequency determinant or transcendental equation is obtained and the zero values of this determinant, found by iteration, correspond to the desired eigenvalues.

4.0 RESULTS

Twenty eigenvalues were found for each of the three problems, five for each set of modes with one, two, three, and four half waves normal to the simple supports. Of each five, three were symmetric and the other two were anti-symmetric in the direction parallel to the simply-supported edges.

For the non-conforming, 12-degree-of-freedom element, n, the number of elements on a side was varied from 2 to 20 in steps of 2. Some larger values of n were chosen in special cases. These results are given in Tables 5, 6, and 7. In a few places, especially for small values of n, it was found impossible to find eigenvalues because the characteristic equations had imaginary roots. Blanks in the Tables indicate these points. It was also impossible to find solutions for the higher modes for small values of n. No great effort was made to find these points by other means, since the results would be very inaccurate in any case.

Since exact solutions for these problems are available, plots of error versus the number of elements are given in Figures 5, 6, and 7. Negative error, that is, solutions below the exact values, are plotted with a solid line, while positive errors are given by dashed lines. It may be seen that most of the error plots tend to asymptote to a straight line as n becomes large. The slope of the straight line is the rate of convergence, and for this element is −2. In a few cases, it can be seen that this slope has not been reached, even though the number of elements on a side is very large.

For the clamped and simply-supported cases, the non-dimensional parameters sometimes start above the exact solutions, cross, and then converge from below. This cross-over point occurs at small values of n. Similar behaviour is observed in the free case, but the cross-over point sometimes occurs at relatively large values of n. Furthermore, for eight modes such cross-overs have not occurred, even though n's as large as 40 are used. It is not known whether or not these cross-overs will ever occur for these modes.

Percentage errors in the solutions found for 10 elements on a side are given in Table 8. With one exception, the errors are all greater than 1%, and often over 10%, even though a large number of elements are used.

Little difference was found between the three-root and the two-root solutions for the clamped and simply-supported cases. Some values changed in the fifth or sixth place. For the free-edge case, the results were significantly different and the two solutions are compared in Table 9 for some of the eigenvalues. As expected, the eigenvalues are nearly the same at large values of n. The three-root solutions are somewhat better at low values of n when the solution is converging from below, but are worse when the solutions are converging from above.

As mentioned previously, Walz, Fulton and Cyrus[12] have shown that the rate of convergence for this element should be n^{-2} for the simply-supported case. They have also given an expression for the error in the solution for large values of n. Figure 8 gives a comparison of the Walz, et al. predictions of error with the actual errors found for a few typical modes. It can be seen that their error estimate is quite good for large values of n (n = 20) for the lower modes, but not as good for the higher ones. Nor are their predictions accurate for smaller values of n.

It may be concluded that the convergence rate, n^{-2}, predicted by Walz, Fulton and Cyrus[12] for the simply-supported case, is verified herein, and that this convergence rate apparently applies to the other two boundary conditions as well (except for some modes in the free two sides case). It must be emphasized, however, that this value only holds for very large values of n and does not apply for array sizes that are readily useable in the direct stiffness method. For these realistic arrays, it is clearly impossible to say anything general about convergence rates.

The results for the three cases found using the 16-degree-of-freedom conforming plate element are given in Tables 10, 11, and 12. Again, 20 eigenvalues are presented for each case and the number of elements on a side, n, was varied from 2 to 10. Solutions for small values of n were more difficult to obtain for this element. Firstly, the characteristic equation often had imaginary roots for small values of n, and this prevented solutions from being obtained. Secondly, the method broke down whenever the number of half waves in the simply-supported direction equalled the number of elements on a side. To complete these Tables, eigenvalues for the 2×2, 3×3 and 4×4 arrays were found using the direct stiffness method. These results are indicated by a star in the Tables. In many places results were found using both methods, and the values always agreed perfectly.

Plots of absolute error versus the number of elements on a side are given in Figures 9, 10, and 11. It is interesting to see that all these curves rapidly approach an asymptote as n increases. The slope of these asymptotes is -4, which is the same as the rate of convergence predicted for the potential energy in static problems.

It is significant that there are slope discontinuities in these error plots for small values of n. This indicates that great care must be taken in extrapolating results calculated for small values of n.

It is particularly surprising to see that the convergence is not monotonic for several modes in the free two sides case. Since this is a conforming element, monotonicity is expected. Closer examination of the requirements for monotonic convergence indicates that finer arrays of element must be contained within coarser arrays to ensure monotonicity; i.e., results from 2×2 and 4×4 arrays should converge monotonically, but results from 2×2 and 3×3 arrays would not necessarily have to converge monotonically.

Percentage errors in the solutions found for 10 elements on a side are given in Table 8. These errors are less than 1% for all but one mode and are especially small for the lower modes. In general, they are all one order of magnitude or more smaller (up to three orders of magnitude in some cases) than the errors found using the non-conforming elements. This Table clearly demonstrates the superiority of the conforming element.

5.0 CONCLUSIONS

A closed form type of analysis using shifting E-operators has been developed to study the dynamic convergence of finite plate-bending elements. Two rectangular elements have been studied, the 12-degree-of-freedom non-conforming element and the 16-degree-of-freedom conforming element. Three dynamic problems involving all three types of boundary conditions have been studied. Exact solutions were found for these problems, so that convergence of the finite element solutions could be considered. Twenty eigenvalues were considered for each of the problems.

The closed form analysis worked well, and large arrays of elements could be studied with little computational effort. It was found that, with few exceptions, the non-conforming element solutions converged from below the exact answers for large values of n, the number of elements on a side, at a rate of n^{-2}. However, the array size needed for such convergence was at least 20×20 or larger, so this convergence rate does not apply for array sizes that are readily useable in the direct stiffness method. The convergence rate of n^{-2} predicted by Walz, Fulton and Cyrus[12)] for square plates simply supported all round was confirmed, but their estimates of the error magnitudes were not good for small values of n.

The conforming element solutions were found to converge to the exact solutions from above at a rate proportional to n^{-4} for values of n larger than 6. This is the same rate as predicted for the convergence of potential energy in static problems. Slope discontinuities in the error plots were found for small values of n, indicating that great care must be exercised in attempting to improve calculated results by extrapolation.

A comparison of the errors involved in using these two elements showed that the conforming element was far superior to the non-conforming element in both magnitude of error and rate of convergence.

6.0 REFERENCES

1. Lindberg, G. M. — Lumped Parameter Methods Applied to Elastic Vibrations. Ph. D. Thesis, Dept. of Engineering, University of Cambridge, England, November 1963.

2. Melosh, R. J. — Basis for Derivation of Matrices for the Direct Stiffness Method. AIAA Journal, Vol. 1, No. 7, July 1963, pp. 1631-1637.

3. Zienkiewicz, O. C., Cheung, Y.K. — The Finite Element Method for Analysis of Elastic Isotropic and Orthotropic Slabs. Proc. Instn. Civ. Engrs., London, Vol. 28, August 1964, pp. 471-488.

4. Dawe, D. J. — A Finite Element Approach to Plate Vibration Problems. J. Mech. Eng. Sci., Vol. 7, No. 1, March 1965, pp. 28-32.

5. Clough, R. W., Tocher, J. L. — Finite Element Stiffness Matrices for Analysis of Plate Bending. IN Matrix Methods in Structural Mechanics. Wright-Patterson Air Force Base, Dayton, Ohio, AFFDL-TR-66-80, 1966, pp. 515-545.

6. Bogner, F. K., Fox, F. L., Schmit, L. A., Jr. — Addendum to: The Generation of Inter-Element Compatible Stiffness and Mass Matrices by the Use of Interpolation Formulas. IN Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Dayton, Ohio, AFFDL-TR-66-80, 1966, pp. 441-443.

7. Butlin, G. A., Leckie, F. A. — A Study of Finite Elements Applied to Plate Flexure. Symposium of Numerical Methods for Vibration Problems, University of Southampton, England, Vol. 3, July 1966, pp. 26-37.

8. Mason, V. — Rectanglar Finite Elements for Analysis of Plate Vibrations. J. Sound Vib., Vol. 7, No. 3, May 1968, pp. 437-448.

9. McLay, R. W. — Completeness and Convergence Properties of Finite Element Displacement Functions - A General Treatment. AIAA Paper, No. 67-143, American Institute of Aeronautics and Astronautics, 5th Aerospace Sci. Mtg. New York, January 23-26, 1967.

10. Tong, P., Pian, T. H. H. — The Convergence of Finite Element Method in Solving Linear Elastic Problems. Int. J. of Solids and Structures, Vol. 3, September 1967, pp. 865-879.

11. Cowper, G. R., Kosko, E., Lindberg, G. M, Olson, M. D. — A High Precision Triangular Plate-Bending Element. NRC, NAE Aero. Report LR-514, National Research Council of Canada, December 1968.

12. Walz, J. E.; Fulton, R. E.; Cyrus, N. J. — Accuracy and Convergence of Finite Element Approximations. Presented at Second Conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Dayton, Ohio, October 15-17, 1968 (to be published).

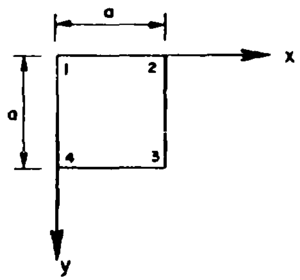
13. Timoshenko, S.; Woinowsky-Krieger, S. — Theory of Plates and Shells. 2nd Edition, McGraw-Hill Book Co., Inc., New York, 1959.

14. Leckie, F. A. — The Application of Transfer Matrices to Plate Vibrations. Ing. Archiv., Vol. 32, 1963, pp. 100-111.

15. Milne-Thomson, L. M. — The Calculus of Finite Differences. Macmillan, 1951.

TABLE 1

STIFFNESS MATRIX, x 45/D, FOR 12-DEGREE-OF-FREEDOM FINITE ELEMENT

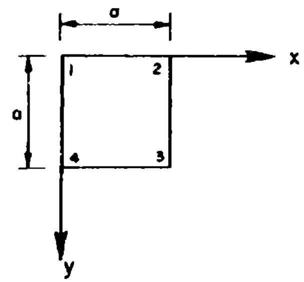


68											
15	68										
111	22	474									
28	0	96	68								
0	22	24	−15	68							
−96	24	−204	−111	111	474						
17	0	39	22	0	−24	68					
0	17	39	0	28	96	15	68				
−39	−39	−66	−24	−96	−204	−111	−111	474			
22	0	24	17	0	−39	28	0	−96	68		
0	28	96	[illegible]	17	39	0	22	−24	−15	68	
24	−96	−204	39	−39	−66	96	−24	−204	111	−111	474

SYMMETRICAL

TABLE 2

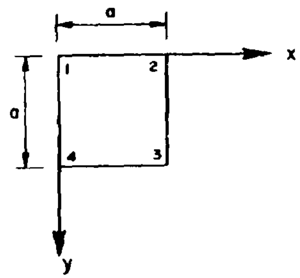
MASS MATRIX, x25200/$\mu\omega^2 a^4$, FOR 12-DEGREE-OF-FREEDOM FINITE ELEMENT



80											
63	80										
461	461	3454									
−60	−42	−274	80								
42	40	199	−63	80		SYMMETRICAL					
274	199	1226	−461	461	3454						
−30	−28	−116	40	−42	−199	80					
−28	−30	−116	42	−60	−274	63	80				
116	116	394	−199	274	1226	−461	−461	3454			
40	42	199	−30	28	116	−60	−42	274	80		
−42	−60	−274	28	−30	−116	42	40	−199	−63	80	
199	274	1226	−116	116	394	−274	−199	1226	461	−461	3454

TABLE 3

STIFFNESS MATRIX, x37800/D, FOR 16-DEGREE-OF-FREEDOM FINITE ELEMENT

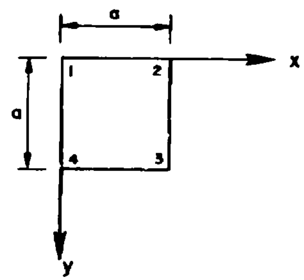


445823															
132191	72575														
132191	39635	72575													
27035	12767	12767	4223												
−219023	−79271	−18791	−6875	445823											
79271	21815	6875	1668	−132191	72575										
−18791	−6875	3023	1512	132191	−39635	72575									
6875	1668	−1512	−695	−27035	12767	12767	4223								
−7775	−34127	−34127	−13283	−219023	18791	−79272	6875	445823							
34127	15983	13283	4211	18791	3023	6875	1512	−132191	72575						
34127	13283	15983	4211	79272	−6875	21815	−1667	−132191	39635	72575					
−13283	−4211	−4211	−995	−6875	−1512	−1667	−695	27035	−12767	−12767	4223				
−219023	−18791	−79272	−6875	−7775	34127	−34127	13283	−219023	79271	18791	−6875	445823			
−18791	3023	−6875	1512	−34127	15983	−13283	4211	−79271	21815	6875	−1668	132191	72575		
79272	6875	21815	1667	34127	−13283	15983	−4211	18791	−6875	3023	−1512	−132191	−39635	72575	
6875	−1512	1667	−695	13283	−4211	4211	−995	6875	−1668	1512	−695	−27035	−12767	12767	4223

SYMMETRICAL

TABLE 4

MASS MATRIX, $\times 176400/\mu\omega^2 a^4$, FOR 16-DEGREE-OF-FREEDOM FINITE ELEMENT



24336															
3431	623														
3431	483	623													
483	87	87	15												
8424	2027	1188	285	24336											
−2027	−468	−285	−65	−3431	623										
1188	285	216	51	3431	−483	623									
−285	−65	−51	−11	−483	87	−87	15								
2916	702	702	168	8424	−1188	2027	−285	24336							
−702	−162	−168	−38	−1188	216	−285	51	−3431	623						
−702	−168	−162	−38	−2027	285	−468	65	−3431	483	623					
168	38	38	9	285	−51	65	−11	483	−87	−87	15				
8424	1188	2027	285	2916	−702	702	−168	8424	−2027	−1188	285	24336			
1188	216	285	51	702	−162	168	−38	2027	−468	−285	65	3431	623		
−2027	−285	−468	−65	−702	168	−162	38	−1188	285	216	−51	−3431	−483	623	
−285	−51	−65	−11	−168	38	−38	9	−285	65	51	−11	−483	−87	87	15

SYMMETRICAL

TABLE 5

NON-DIMENSIONAL FREQUENCY PARAMETERS FOR SQUARE PLATE CLAMPED TWO SIDES NON-CONFORMING SOLUTIONS

Number of Elements On A Side	MODE 1-1	1-2	1-3	1-4	1-5
2	663.013	1864.2			
4	768.086	4451.084	15806.14	56744.06	
6	803.504	4599.860	16138.05	43020.39	
8	817.895	4678.053	16277.32	42789.45	94457.18
10	824.950	4720.304	16385.91	42856.03	93839.52
12	828.893	4745.030	16458.84	42962.95	93752.92
14	831.308	4760.575	16507.94	43054.63	93807.35
16	832.892	4770.926	16541.98	43125.81	93889.80
20	834.770	4783.371	16584.26	43221.83	94037.19
Exact Solutions	838.1518	4806.235	16665.66	43427.11	94443.215

One Half Wave Normal to Simple Supports

Number of Elements On A Side	MODE 3-1	3-2	3-3	3-4	3-5
2					
4	9699.55				
6	9872.23	16816.67	32458.23	63198.99	114953.66
8	10061.64	17752.40	34677.59	67100.86	124215.39
10	10180.05	18330.77	36177.69	69996.92	128696.45
12	10253.69	18691.85	37156.74	72014.67	132094.37
14	10301.49	18927.34	37811.99	73417.38	134591.61
16	10333.94	19087.82	38265.91	74412.63	136425.87
20	10373.58	19284.62	38830.65	75676.91	138826.16
Exact Solutions	10448.15	19657.30	39924.25	78204.81	143844.55

Three Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 2-1	2-2	2-3	2-4	2-5
2					
4	2640.983	7218.699	19137.02		
6	2798.470	7906.509	21012.76	49132.88	99736.99
8	2875.821	8289.526	21980.48	50736.26	104351.65
10	2916.377	8501.859	22576.26	51873.58	105775.67
12	2939.779	8627.979	22949.74	52660.58	106991.53
14	2954.378	8707.990	23194.10	53204.12	107927.64
16	2964.055	8761.592	23360.98	53587.66	108629.33
20	2975.641	8826.371	23566.02	54072.06	109559.12
Exact Solutions	2996.804	8946.375	23955.52	55030.33	111524.73

Two Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 4-1	4-2	4-3	4-4	4-5
2					
4		34274.79			
6	28119.59	37079.04	56115.62		
8	28216.10	38556.38	59506.91	96913.89	159454.65
10	28401.92	39690.94	62316.00	102071.39	167352.22
12	28548.12	40462.34	64288.43	105935.10	173748.46
14	28653.26	40988.49	65659.99	108717.61	178574.81
16	28728.75	41356.82	66632.44	110734.08	182175.46
20	28825.15	41818.86	67866.53	113342.60	186952.64
Exact Solutions	29017.92	42723.69	70329.12	118706.33	197168.52

Four Half Waves Normal to Simple Supports

TABLE 6

NON-DIMENSIONAL FREQUENCY PARAMETERS FOR SQUARE PLATE SIMPLY SUPPORTED ALL ROUND
NON-CONFORMING SOLUTIONS

Number of Elements On A Side	MODE 1-1	1-2	1-3	1-4	1-5
2	322.626				
4	366.100	2241.620	9231.57		77545.39
6	378.428	2332.795	9372.99	27567.68	66529.53
8	383.173	2373.835	9496.56	27602.53	65330.00
10	385.452	2394.744	9572.40	27724.42	65179.50
12	386.712	2406.650	9619.06	27824.10	65243.49
14	387.479	2414.022	9649.19	27896.67	65338.18
16	387.981	2418.887	9669.57	27949.07	65423.89
20	388.574	2424.690	9694.40	28016.26	65550.44
Exact Solutions	389.6363	2435.227	9740.909	28151.22	65848.54

One Half Wave Normal to Simple Supports

Number of Elements On A Side	MODE 3-1	3-2	3-3	3-4	3-5
2	SAME AS MODE 1-3	SAME AS MODE 2-3			
4				57761.33	
6			26132.72	49526.00	92967.71
8			27857.76	52597.47	97160.19
10			28956.86	54862.44	100890.26
12			29654.12	56390.30	103643.37
14			30113.42	57431.50	105612.15
16			30428.50	58161.03	107032.79
20			30817.23	59077.73	108863.56
Exact Solutions	9740.909	16462.13	31560.54	60880.68	112604.91

Three Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 2-1	2-2	2-3	2-4	2-5
2	SAME AS MODE 1-2				
4		5162.019	13445.49		
6		5623.970	14500.88	34582.36	74859.00
8		5857.604	15186.35	35864.94	76142.05
10		5982.219	15585.93	36746.34	77482.53
12		6054.848	15829.22	37324.70	78513.36
14		6100.434	15985.78	37712.51	79259.75
16		6130.772	16091.61	37981.35	79800.52
20		6167.[illegible]26	16220.50	38315.91	80498.25
Exact Solutions	2435.227	623[illegible].182	16462.13	38963.64	81921.04

Two Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 4-1	4-2	4-3	4-4	4-5
2	SAME AS MODE 1-4	SAME AS MODE 2-4	SAME AS MODE 3-4		
4					
6					
8				82592.30	133258.43
10				86890.65	139985.61
12				89983.52	145295.60
14				92162.95	149210.22
16				93721.66	152086.66
20				95715.50	155852.62
Exact Solutions	28151.22	38963.64	60880.68	99746.91	163744.68

Four Half Waves Normal to Simple Supports

TABLE 7

NON-DIMENSIONAL FREQUENCY PARAMETERS FOR SQUARE PLATE FREE TWO SIDES NON-CONFORMING SOLUTIONS

Number of Elements On A Side	MODE 1-1	1-2	1-3	1-4	1-5
2	109.225		1591.612		
4	95.3118	264.135	1314.582	5476.430	17164.03
6	92.9870	256.118	1311.480	5498.602	17471.75
8	92.2792	253.939	1314.903	5536.744	17530.12
10	91.9823	253.116	1317.733	5562.331	17598.02
12	91.8323	252.736	1319.726	5578.974	17649.46
14	91.7466	252.538	1321.126	5590.153	17686.39
16	91.6934	252.423	1322.130	5597.951	17713.09
20	91.6336	252.306	1323.430	5607.766	17747.59
24	91.6027	252.253			
30	91.5784	252.215			
40	91.5606	252.191			
Exact Solutions	91.5387	252.1715	1326.219	5627.702	17819.90

One Half Wave Normal to Simple Supports

Number of Elements On A Side	MODE 3-1	3-2	3-3	3-4	3-5
2					
4	8548.342				
6	8213.514	9986.156	15140.12	26102.22	47265.68
8	8005.068	9624.656	14872.62	26000.39	47020.57
10	7891.363	9425.762	14755.54	26107.90	47565.94
12	7827.386	9316.883	14707.29	26230.29	48070.70
14	7789.262	9253.712	14687.82	26334.09	48462.43
16	7765.183	9214.792	14680.74	26416.82	48758.69
20	7738.125	9172.396	14680.00	26533.56	49157.78
24	7724.316	9151.594	14684.14		
30	7713.715	9136.256	14690.72		
40	7706.085	9125.772	14698.48		
Exact Solutions	7697.652	9115.480	14714.02	26823.64	50066.07

Three Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 2-1	2-2	2-3	2-4	2-5
2					
4	1653.296	2435.918	4722.986		27654.72
6	1571.898	2269.273	4944.046	11734.13	26937.47
8	1540.192	2206.858	4902.671	11804.88	27196.69
10	1526.022	2180.405	4895.390	11890.26	27511.23
12	1518.745	2167.457	4896.348	11955.11	27745.91
14	1514.590	2160.383	4899.196	12002.13	27913.64
16	1512.021	2156.183	4902.196	12036.51	28034.86
20	1509.169	2151.754	4907.176	12081.64	28192.03
24	1507.720	2149.645			
30	1506.602	2148.122			
40	1505.789	2147.104			
Exact Solutions	1504.861	2146.139	4922.187	12180.20	28526.84

Two Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 4-1	4-2	4-3	4-4	4-5
2					
4					
6	26338.17	29701.50	37819.63	54017.19	
8	25717.49	28959.79	37476.18	53960.62	82518.54
10	25320.35	28308.24	37048.38	54040.62	83236.12
12	25064.31	27874.60	36785.99	54177.25	84071.86
14	24898.23	27596.50	36639.04	54324.49	84781.79
16	24787.64	27414.97	36556.74	54459.26	85347.96
20	24658.39	27207.88	36482.89	54672.51	86146.76
24	24950.62	27102.47	36458.36		
30	24538.01	27022.97	36450.19		
40	24500.05	26967.69	36454.13		
Exact Solutions	24458.86	26912.55	36481.95	55310.25	88119.88

Four Half Waves Normal to Simple Supports

TABLE 8

PERCENTAGE ERRORS IN FINITE ELEMENT SOLUTIONS, n = 10

Percentage Errors in Non-Dimensional Parameter, $\gamma = \mu\omega^2 L^4/D$, for 10 elements on a side

Mode γ_{ij}	Simply Supported All Round		Clamped Two Sides		Free Two Sides	
i-j	Non-Conforming	Conforming	Non-Conforming	Conforming	Non-Conforming	Conforming
1-1	− 1.074	.0007	− 1.575	.00659	.484	.0016
1-2	− 1.662	.014	− 1.788	.044	.374	.0021
1-3	− 1.730	.087	− 1.679	.176	− .640	.011
1-4	− 1.516	.295	− 1.315	.491	− 1.162	.050
1-5	− 1.016	.734	− 0.639	1.092	− 1.245	.175
2-1	− 1.662	.014	− 2.684	.023	1.406	.024
2-2	− 4.042	.011	− 4.969	.043	1.597	.021
2-3	− 5.323	.054	− 5.758	.142	− .544	.025
2-4	− 5.691	.217	− 5.736	.410	− 2.385	.066
2-5	− 5.418	.597	− 5.155	.956	− 3.560	.168
3-1	− 1.730	.087	− 2.566	.097	2.516	.114
3-2	− 5.323	.054	− 6.748	.092	3.404	.105
3-3	− 8.250	.056	− 9.384	.146	.282	.086
3-4	− 9.885	.156	−10.495	.347	− 2.668	.102
3-5	−10.403	.450	−10.531	.808	− 4.994	.185
4-1	− 1.516	.295	− 2.123	.305	3.522	.348
4-2	− 5.691	.217	− 7.098	.256	5.186	.331
4-3	− 9.885	.156	−11.394	.249	1.553	.269
4-4	−12.889	.177	−14.014	.367	− 2.295	.235
4-5	−16.352	.367	−15.122	.720	− 5.542	.270

TABLE 9

COMPARISON OF THREE-ROOT AND TWO-ROOT SOLUTIONS FOR PLATE FREE TWO SIDES NON-CONFORMING SOLUTIONS

	MODE							
	1-1		1-2		1-3		1-4	
Number of Elements On A Side	2-Root Solutions	3-Root Solutions	2-Root Solutions	3-Root Solutions	2-Root Solutions	3-Root Solutions	2-Root Solutions	3-Root Solutions
4	93.2634	95.3118	251.547	264.135	1253.775	1314.582	5242.435	5476.430
8	92.0099	92.2792	252.108	253.939	1305.262	1314.903	5500.016	5536.744
12	91.7518	91.8323	252.134	252.736	1316.566	1319.726	5566.646	5578.974
16	91.6593	91.6934	252.140	252.423	1320.698	1322.130	5592.422	5597.951
20	91.6161	91.6336	252.144	252.306	1322.647	1323.430	5604.833	5607.766
Exact Solutions	91.5387		252.172		1326.219		5627.702	

	MODE							
	1-5		2-2		3-3		4-4	
Number of Elements On A Side	2-Root Solutions	3-Root Solutions	2-Root Solutions	3-Root Solutions	2-Root Solutions	3-Root Solutions	2-Root Solutions	3-Root Solutions
4	16336.20	17164.03	2143.410	2435.918				
8	17422.94	17530.12	2153.154	2206.858	14143.19	14872.62	50095.87	53960.62
12	17612.42	17649.46	2150.476	2167.457	14452.37	14707.29	52648.24	54177.25
16	17696.05	17713.09	2148.844	2156.183	14565.06	14680.74	53742.52	54459.26
20	17738.38	17747.59	2147.950	2151.754	14617.72	14680.00	54285.71	54672.51
Exact Solutions	17819.90		2146.139		14714.02		55310.25	

TABLE 10

NON-DIMENSIONAL FREQUENCY PARAMETERS FOR SQUARE PLATE CLAMPED TWO SIDES CONFORMING SOLUTIONS

Number of Elements On A Side	MODE 1-1	1-2	1-3	1-4	1-5
2	870.285	7923.934*			
3	844.666	5003.396*	23690.00*	89064.49*	
4	840.234	4884.959	17381.45	58360.90	155422.89
5	839.014	4839.212	17093.17	45324.57	123251.56
6	838.571	4822.244	16883.13	44874.96	98589.45
7	838.379	4814.906	16785.37	44268.14	98167.39
8	838.286	4811.330	16736.56	43934.66	96824.60
10	838.207	4808.328	16695.02	43640.50	95474.65
Exact Solutions	838.152	4806.235	16665.66	43127.11	94443.22

One Half Wave Normal to Simple Supports

Number of Elements On A Side	MODE 2-1	2-2	2-3	2-4	2-5
2	3470.769*	13120.00*			
3	3061.051	9343.560*	31772.81*	10386[illegible].40*	
4	3019.273	9084.430	24963.16	70896.79*	175030.42*
5	3006.594	9004.090	24474.89	57357.83	141470.17
6	3001.717	8974.652	24212.92	56625.60	116269.45
7	2999.528	8961.850	24095.54	55939.17	115473.16
8	2998.431	8955.551	24037.92	55573.27	114016.37
10	2997.486	8950.197	23989.44	55256.12	112591.32
Exact Solutions	2996.804	8946.375	23955.52	55030.34	111524.73

Two Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 3-1	3-2	3-3	3-4	3-5
2	15083.67*	29406.78*			
3	12387.33*	22276.40*	51119.02*	134700.30*	
4	10781.34*	20223.31*	41779.14	96030.29	
5	10594.20	19905.38	40788.98	81437.13	
6	10521.44	19782.79	40352.37	80183.14	149698.13
7	10488.76	19727.53	40158.83	79312.22	148276.54
8	10472.39	19699.62	40063.47	78861.55	146596.18
10	10458.31	19675.31	39982.45	78476.40	145006.62
Exact Solutions	10448.15	19657.30	39924.25	78204.81	143844.55

Three Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 4-1	4-2	4-3	4-4	4-5
2					
3	37990.20*	56069.55*	91222.1*		
4	34939.06*	49193.54*	78879.08*	145099.40*	278137.76*
5				124302.30	233386.07
6	29645.76	43479.40	71547.92		
7	29368.45	43148.88	71010.41	120456.07	202692.66
8	29228.07	42980.47	70739.51	119748.28	200552.46
10	29106.42	42833.13	70504.20	119141.93	198587.41
Exact Solutions	29017.92	42723.69	70329.12	118706.33	197168.52

Four Half Waves Normal to Simple Supports

*Values found using direct stiffness method.

TABLE 11

NON-DIMENSIONAL FREQUENCY PARAMETERS FOR SQUARE PLATE SIMPLY SUPPORTED ALL ROUND CONFORMING SOLUTIONS

Number of Elements On A Side	MODE 1-1	1-2	1-3	1-4	1-5
2	391.318*	2808.01*	14190.0*	43735.0*	
3	389.967	2473.778	11594.16*	36918.45	113091.4*
4	389.740	2447.900	10037.31	32229.2*	82687.4
5	389.679	2440.503	9868.48	29326.71	
6	389.657	2437.793	9803.98	28749.55	69149.41
7	389.647	2436.619	9775.46	28483.57	67734.91
8	389.643	2436.046	9761.36	28349.63	66988.59
10	389.639	2435.564	9749.38	28234.26	66331.90
Exact Solutions	389.6363	2435.227	9740.909	28151.22	65848.54

One Half Wave Normal to Simple Supports

Number of Elements On A Side	MODE 3-1	3-2	3-3	3-4	3-5
2	SAME AS MODE 1-3	SAME AS MODE 2-3	42028.0*		
3			35640.0*	72638.3*	
4			33975.8*	67398.7*	130704.1*
5			31841.53	62303.94	
6			31696.77	61591.45	116175.1
7			31634.03	61270.08	114622.3
8			31603.53	61110.60	113814.8
10			31578.07	60975.42	113112.0
Exact Solutions	9740.909	16462.13	31560.54	60880.68	112604.91

Three Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 2-1	2-2	2-3	2-4	2-5
2	SAME AS MODE 1-2	7040.00*	21591.00*		
3		6318.06*	18432.02*	48078.91	130816.3*
4		6261.09	16788.35	42928.19*	99096.0
5		6245.185	16600.61	40183.64	
6		6239.471	16529.92	39580.44	85299.85
7		6237.027	16499.00	39304.59	83843.70
8		6235.85	16483.84	39166.40	83079.70
10		6234.860	16471.06	39048.04	82410.21
Exact Solutions	2435.227	6234.182	16462.13	38963.64	81921.04

Two Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 4-1	4-2	4-3	4-4	4-5
2	SAME AS MODE 1-4	SAME AS MODE 2-4	SAME AS MODE 3-4		
3					
4				112640.0*	
5				102442.96	
6					
7				100479.2	166180.0
8				100177.5	165194.7
10				99923.0	164345.7
Exact Solutions	28151.22	38963.64	60880.68	99746.91	163744.68

Four Half Waves Normal to Simple Supports

*Values found using direct stiffness method.

TABLE 12

NON-DIMENSIONAL FREQUENCY PARAMETERS FOR SQUARE PLATE FREE TWO SIDES CONFORMING SOLUTIONS

Number of Elements On A Side	MODE 1-1	1-2	1-3	1-4	1-5
2	92.3723	254.534	1336.617	6792.605*	34391.96*
3	91.7117	252.734	1336.960	5666.684	21756.27
4	91.5945	252.363	1330.572	5694.700	18036.08
5	91.5618	252.2528	1328.193	5662.926	18137.83
6	91.5499	252.2115	1327.224	5646.658	18011.44
7	91.5448	252.1933	1326.779	5638.573	17934.56
8	91.5423	252.1844	1326.554	5634.319	17891.28
10	91.5402	252.1768	1326.360	5630.53	17851.12
Exact Solutions	91.5387	252.1715	1326.219	5627.702	17819.90

One Half Wave Normal to Simple Supports

Number of Elements On A Side	MODE 3-1	3-2	3-3	3-4	3-5
2	11963.017*	13545.01*	19617.55*	33736.66*	
3	9539.736*	11009.70*	16803.34*	29129.01*	56999.56*
4	7995.091*	9432.922*	15108.43*	27496.85*	50807.37
5	7826.979	9254.304	14889.83	27155.83	50951.78
6	7762.075	9184.968	14803.21	27001.66	50602.82
7	7733.142	9153.904	14763.82	26926.42	50393.38
8	7718.743	9138.380	14743.92	26886.73	50273.22
10	7706.437	9125.056	14726.65	26851.01	50158.88
Exact Solutions	7697.652	9115.480	14714.02	26823.64	50066.07

Three Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 2-1	2-2	2-3	2-4	2-5
2	1869.679*	2537.978*	5396.798*	13973.81*	
3	1543.174	2191.854	5019.373	12317.40*	32767.25
4	1517.720	2161.799	4959.868	12347.14	28770.76
5	1510.285	2152.834	4939.291	12269.15	28985.31
6	1507.521	2149.452	4930.947	12229.11	28809.77
7	1506.312	2147.958	4927.101	12208.85	28698.85
8	1505.718	2147.217	4925.143	12198.04	28635.05
10	1505.215	2146.587	4923.437	12188.26	28574.79
Exact Solutions	1504.861	2146.139	4922.187	12180.20	28526.84

Two Half Waves Normal to Simple Supports

Number of Elements On A Side	MODE 4-1	4-2	4-3	4-4	4-5
2	39966.44*	42781.88*			
3	33185.95*	35825.32	45921.20	65466.29	103984.04
4	30273.97*	32817.44*	42635.81*	62112.18*	95295.56*
5					90659.24
6	25064.17	27541.36	37167.69	56182.73	89559.98
7	24796.72	27264.40	36866.50	55807.23	88976.81
8	24661.34	27123.86	36713.33	55612.72	88656.29
10	24544.07	27001.76	36579.92	55440.31	88358.36
Exact Solutions	24458.86	26912.55	36481.95	55310.25	88119.88

Four Half Waves Normal to Simple Supports

*Values found using direct stiffness method.

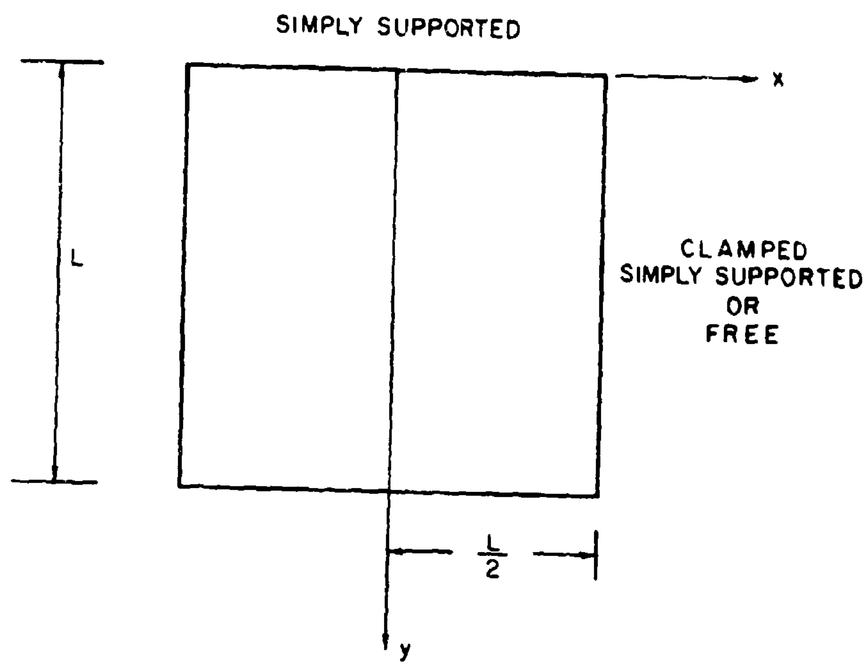


FIG. 1 : CO-ORDINATE SYSTEM FOR PROBLEMS CONSIDERED

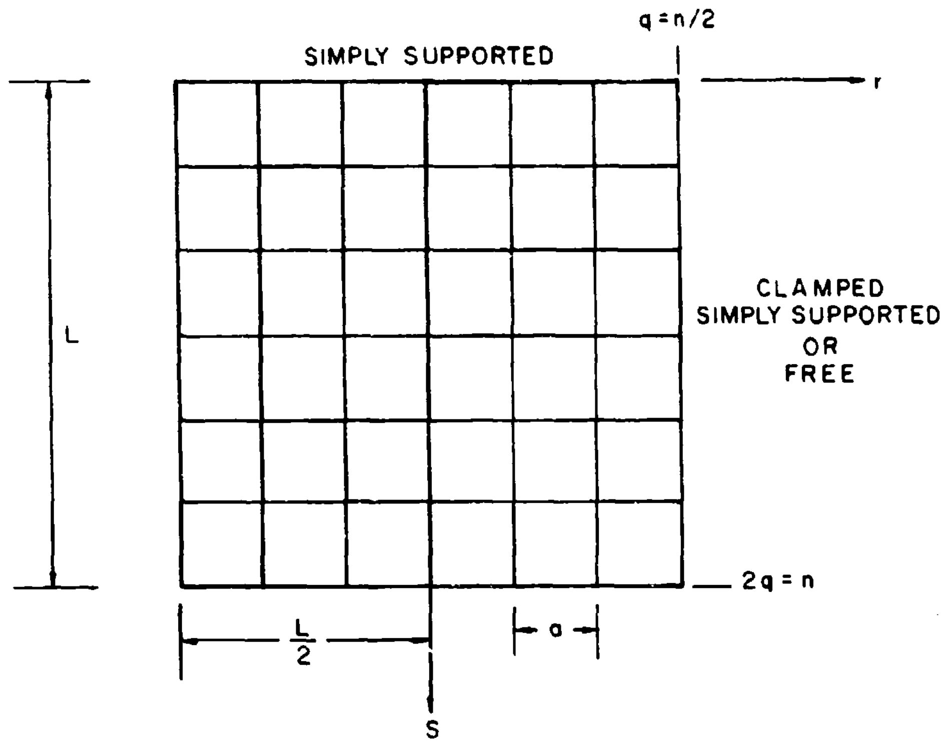


FIG. 2: FINITE ELEMENT ARRAY FOR PROBLEMS CONSIDERED

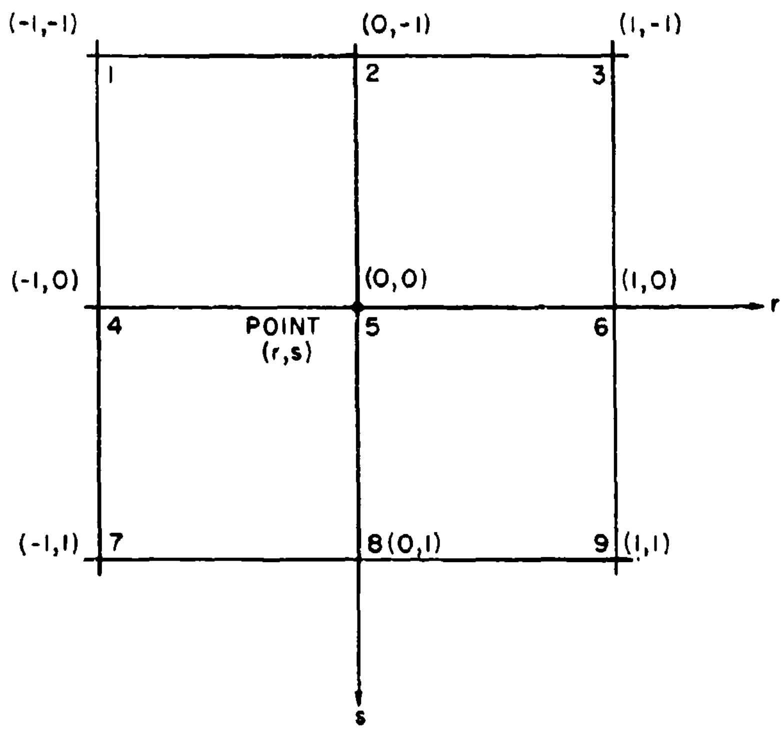


FIG. 3: FINITE ELEMENT REPRESENTATION OF AN INTERNAL POINT

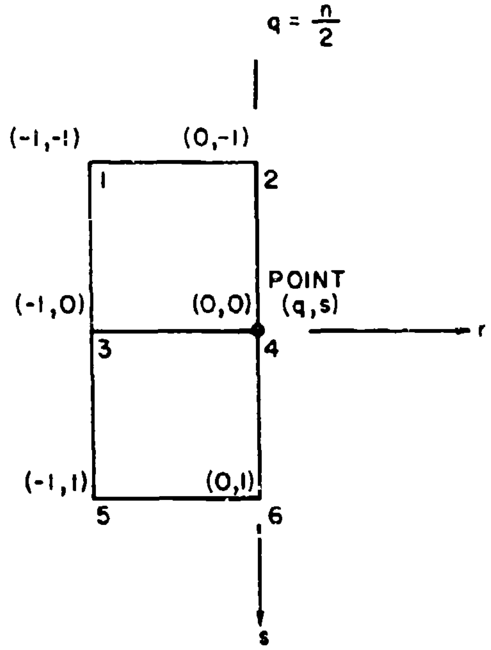


FIG.4: FINITE ELEMENT REPRESENTATION OF AN EDGE POINT

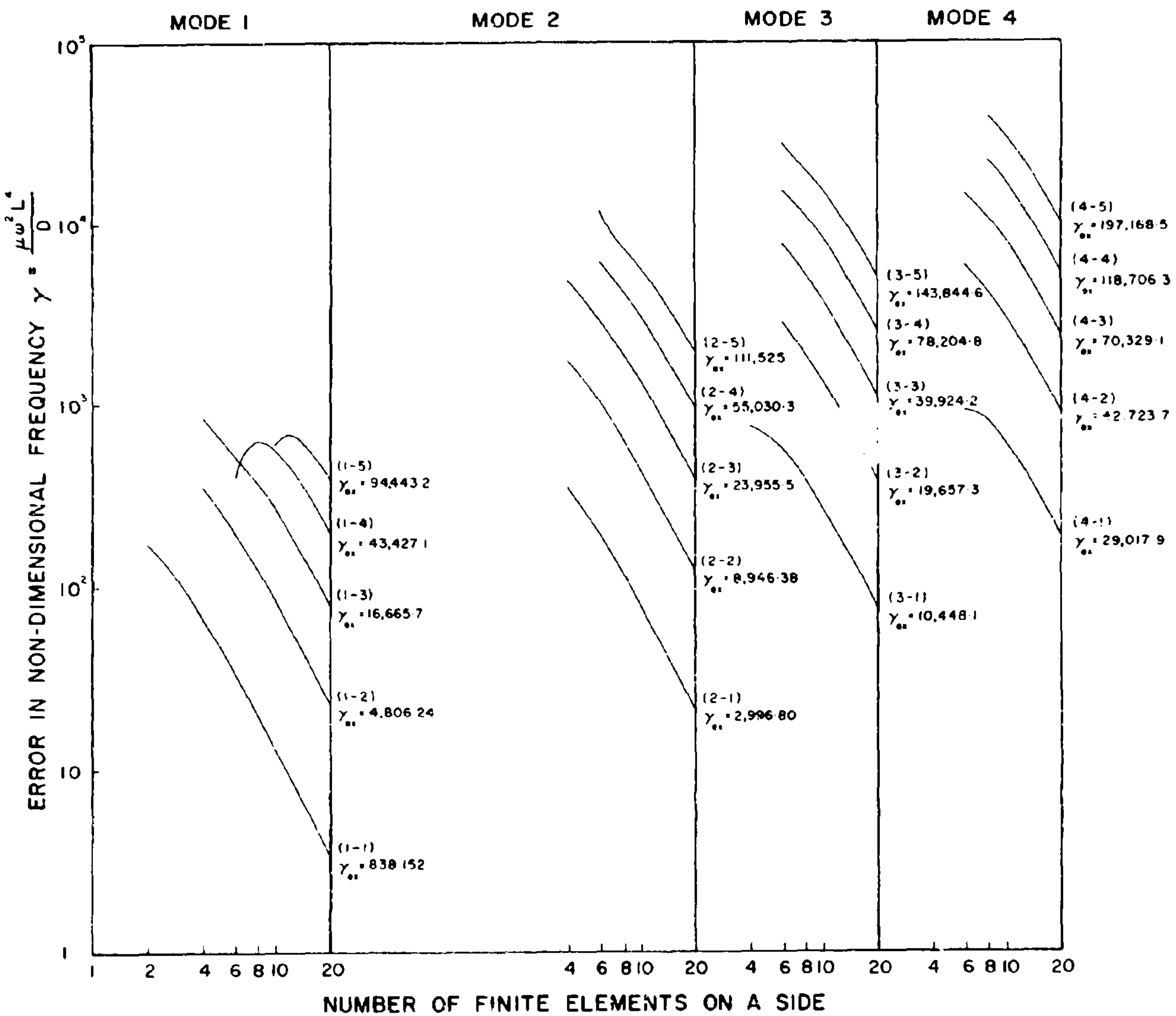


FIG. 5 : ERRORS IN NON-CONFORMING SOLUTIONS FOR PLATE CLAMPED TWO SIDES

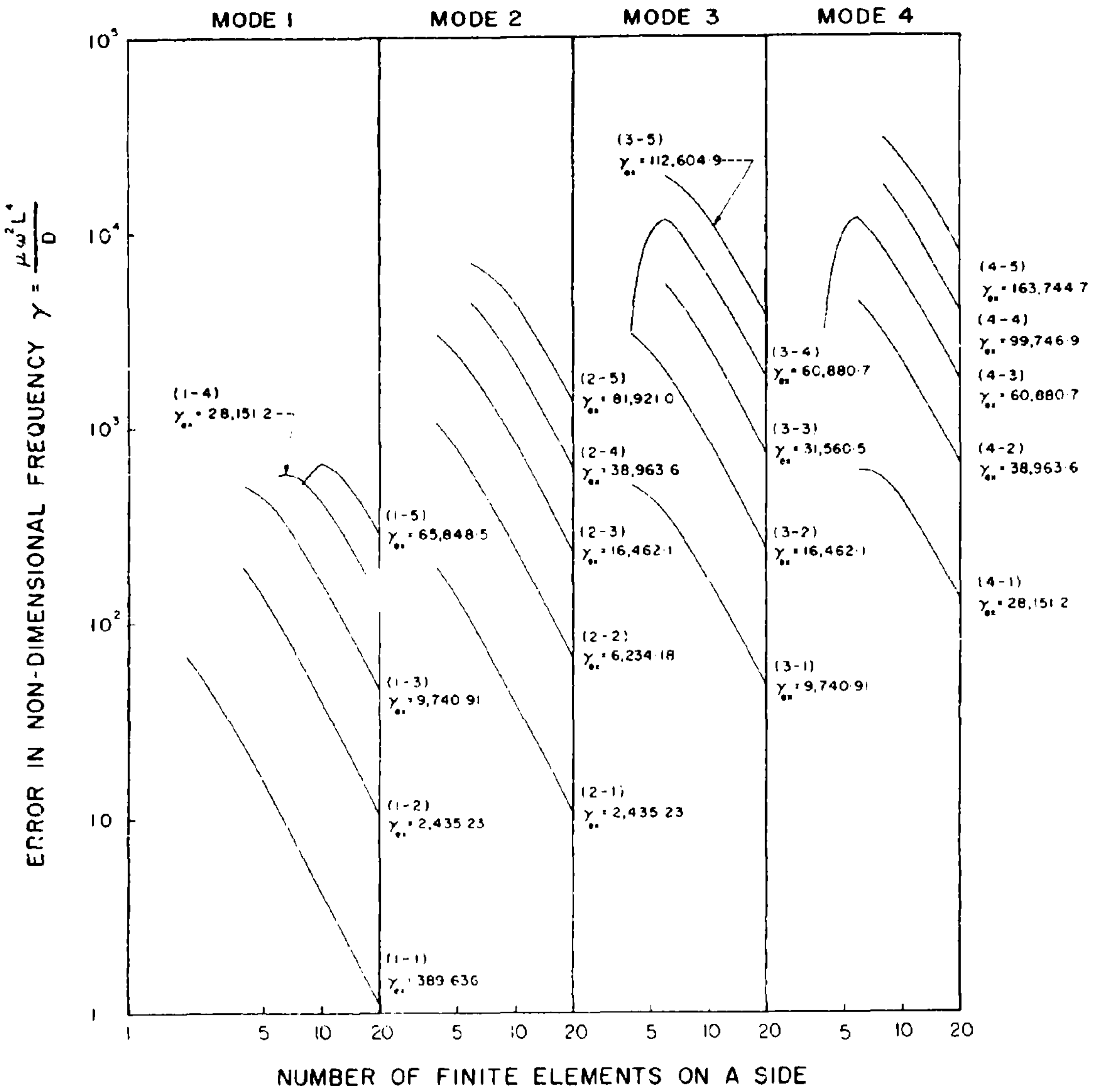


FIG. 6 : ERRORS IN NON-CONFORMING SOLUTIONS FOR PLATE SIMPLY SUPPORTED ALL ROUND

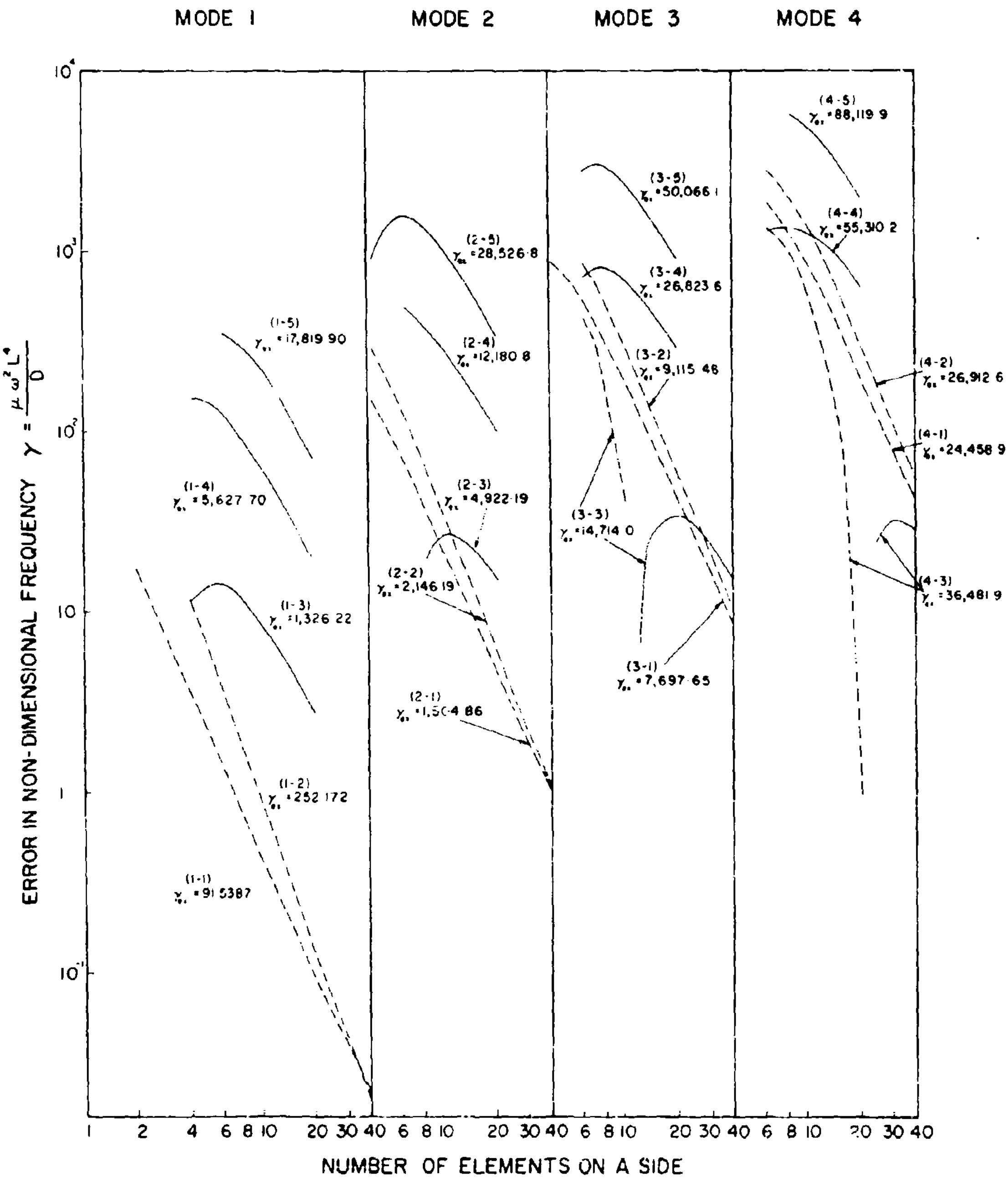


FIG. 7: ERRORS IN NON-CONFORMING SOLUTIONS FOR PLATE FREE TWO SIDES

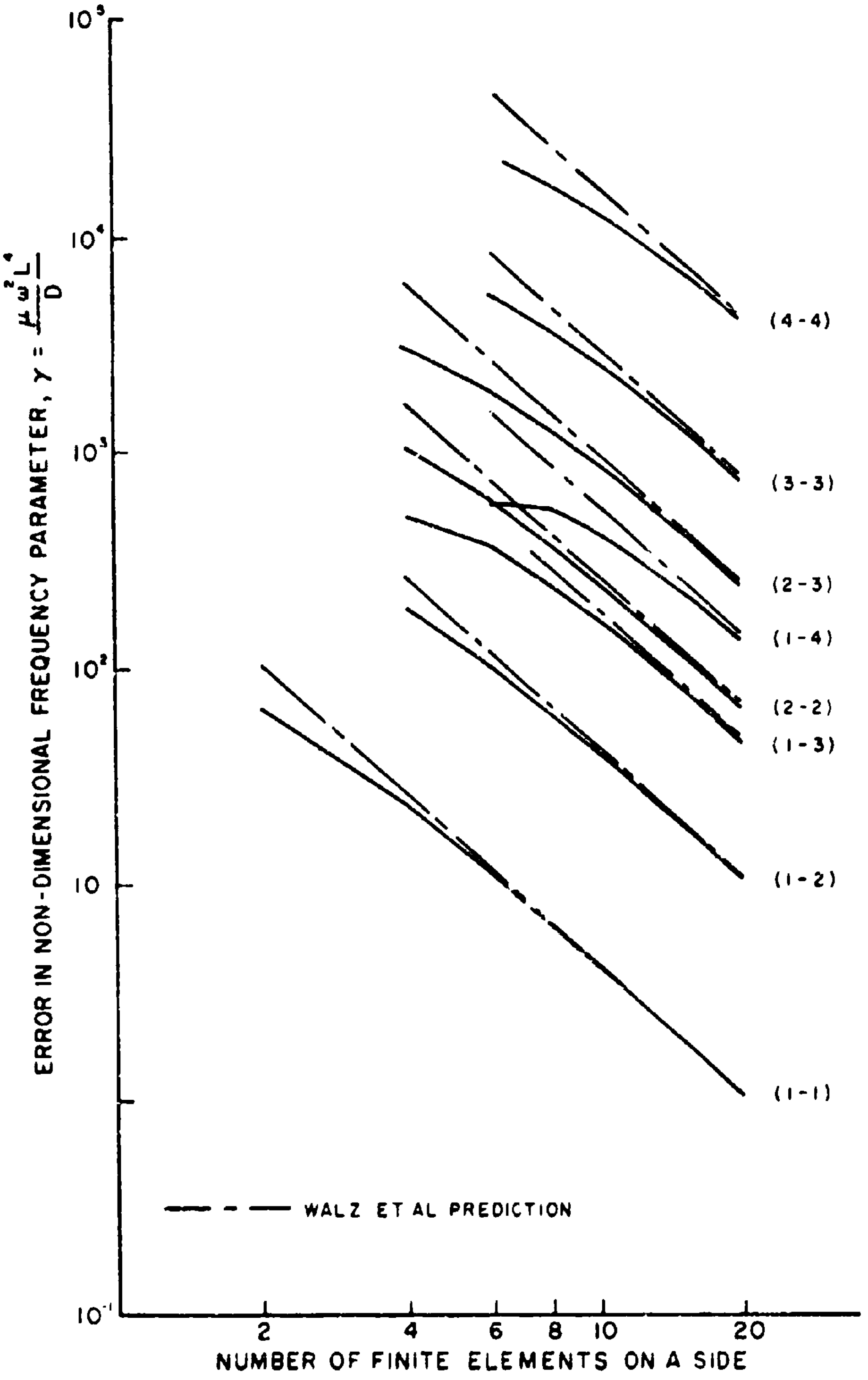


FIG. 8: COMPARISON OF WALTZ, ET AL PREDICTIONS WITH ACTUAL ERRORS

PLATE SIMPLY SUPPORTED ALL ROUND

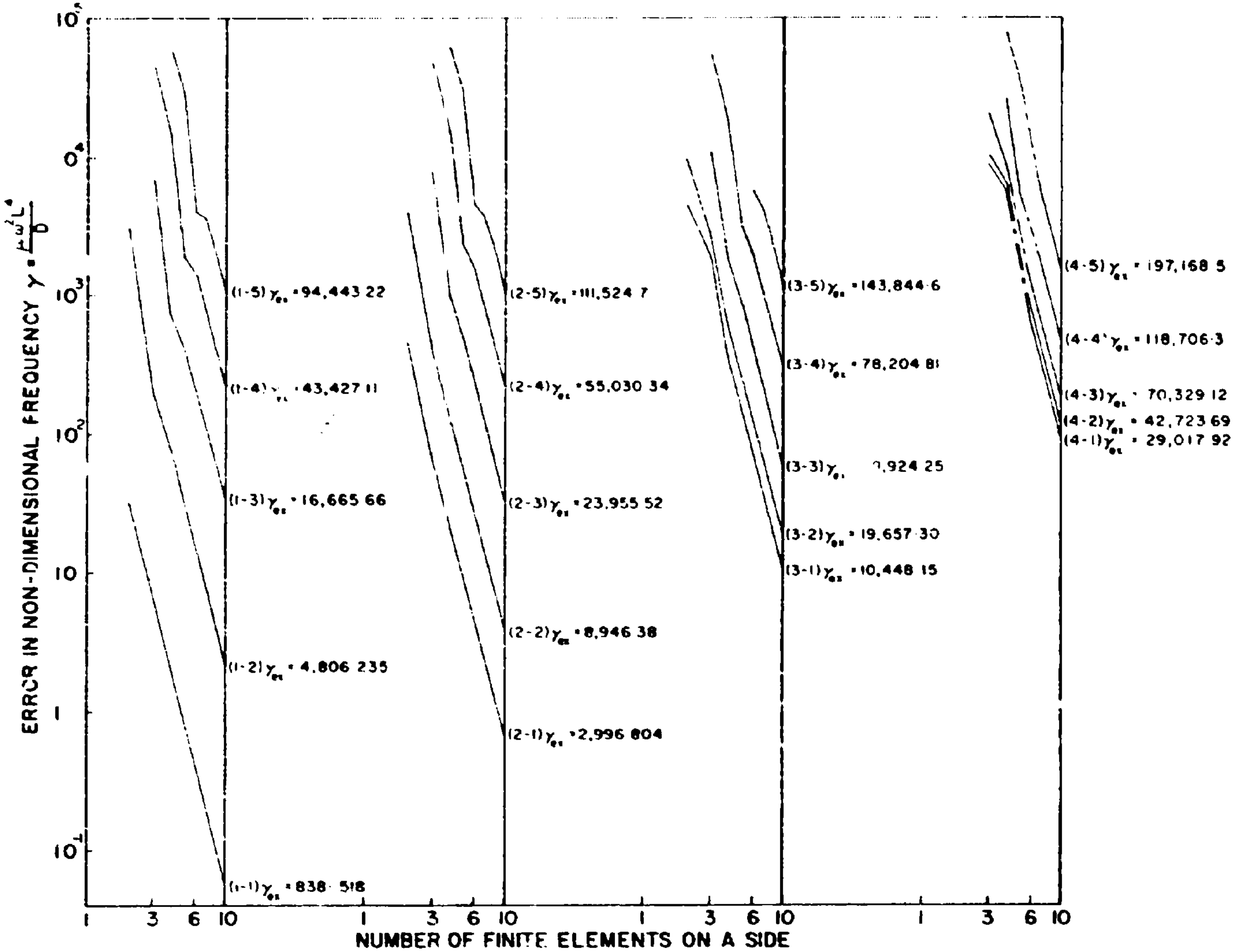


FIG.9: ERRORS IN CONFORMING SOLUTIONS FOR PLATE CLAMPED TWO SIDES

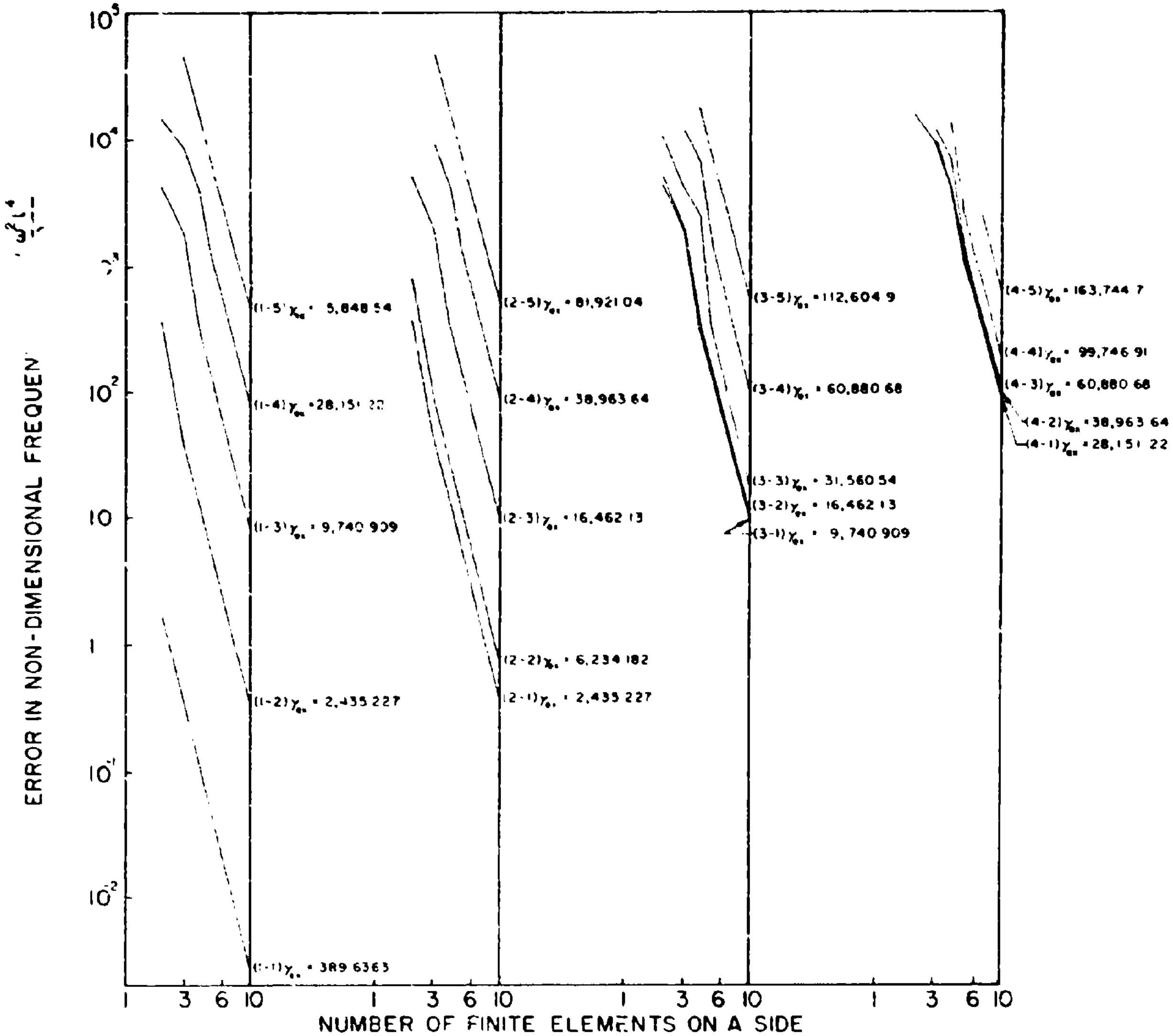


FIG. 10: ERRORS IN CONFORMING SOLUTIONS FOR PLATE SIMPLY SUPPORTED ALL ROUND

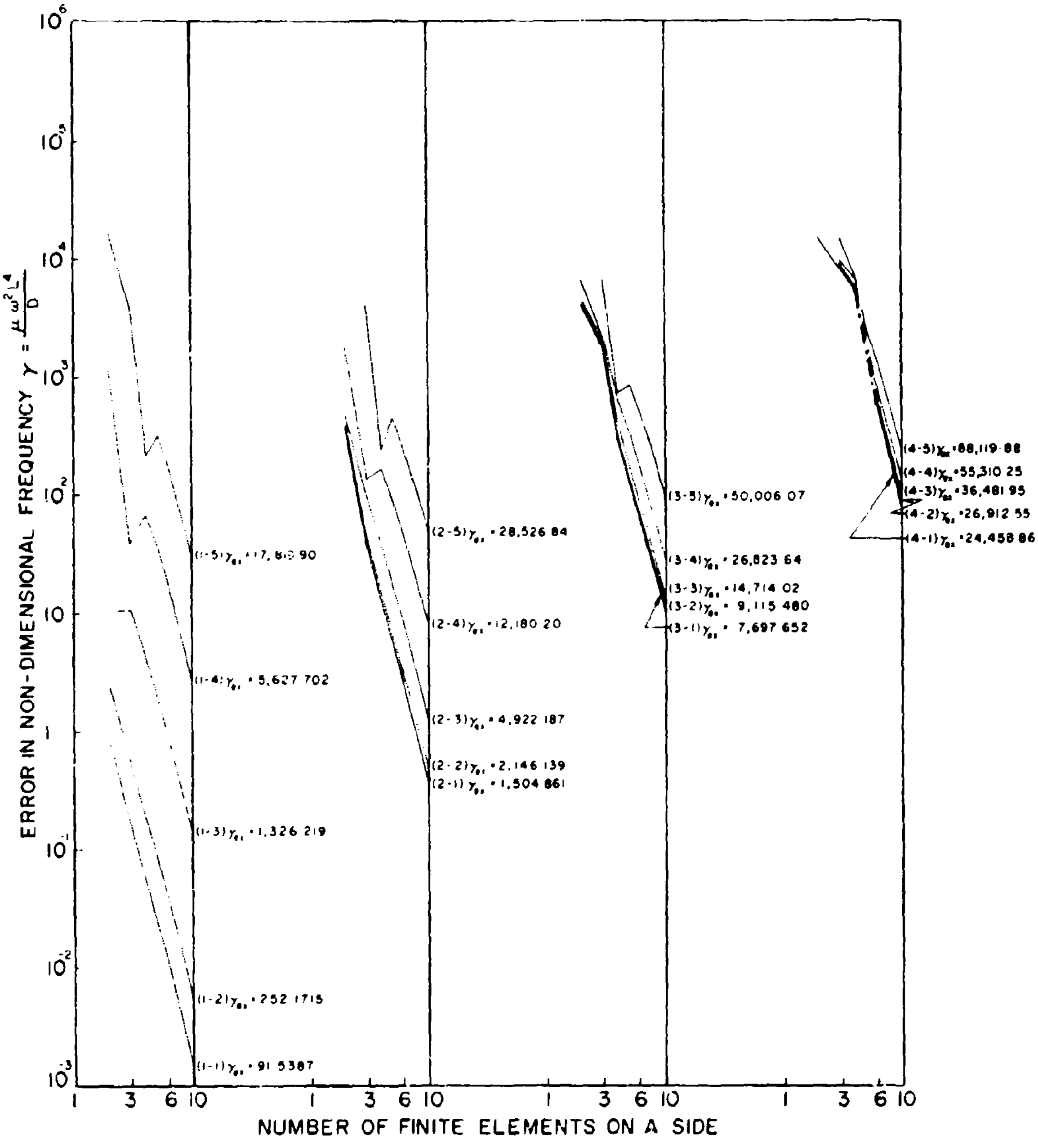


FIG. 11: ERRORS IN CONFORMING SOLUTIONS FOR A PLATE FREE TWO SIDES

APPENDIX A
FINITE DIFFERENCE SHIFTING E-OPERATORS
(see Reference 15)

In one-dimensional space, r, these operators are defined as

$$E_r^k\, w_r = w_{r+k}$$

and provide a concise notation for the function, w_r, specified at finite points. In two-dimensional space, r and s, the operators are

$$E_r^k\, E_s^m\, w_{r,s} = w_{r+k,\,s+m}$$

These operators obey the rule

$$F_1(E_r)\, F_2(E_s)\, a^{kr}\, b^{ms} = a^{kr}\, b^{ms}\, F_1(a^k)\, F_2(b^m)$$

Using this rule, finite difference or finite element equations may be transformed into ordinary exponential equations. For example,

$$(E_r^{-1} - E_r^{1})\sin(\pi r/R) = (E_r^{-1} - E_r^{1})\left[-i(e^{i\pi r/R} - e^{-i\pi r/R})/2\right]$$

$$= -i\left[(e^{-i\pi/R} - e^{i\pi/R})\, e^{i\pi r/R} - (e^{i\pi/R} - e^{-i\pi/R})\, e^{-i\pi r/R}\right]/2$$

$$= -2\sin(\pi/R)\cos(\pi r/R)$$

A list of the transformations needed in the analysis are given below.

SHIFTING E-OPERATOR TRANSFORMATIONS k is any integer

$$(E_p^{-k} - E_p^{k})\, e^{\sigma p} = -2\sinh k\sigma\, e^{\sigma p}$$

$$(E_p^{-k} + E_p^{k})\, e^{\sigma p} = 2\cosh k\sigma\, e^{\sigma p}$$

$$(E_p^{-k} - E_p^{k})\, e^{\sigma p} = -2\, i \sin k\sigma\, e^{\sigma p}$$

$$(E_p^{-k} + E_p^{k})\, e^{\sigma p} = 2\cos k\sigma\, e^{\sigma p}$$

$$(E_p^{-k} - E_p^{k})\sinh \sigma p = -2\sinh k\sigma \cosh \sigma p$$

$$(E_p^{-k} + E_p^{k})\sinh \sigma p = 2\cosh k\sigma \sinh \sigma p$$

$$(E_p^{-k} - E_p^{k})\cosh \sigma p = -2\sinh k\sigma \sinh \sigma p$$

$$(E_p^{-k} + E_p^{k})\cosh \sigma p = 2\cosh k\sigma \cosh \sigma p$$

$$(E_p^{-k} - E_p^{k})\sin \sigma p = -2\sin k\sigma \cos \sigma p$$

$$(E_p^{-k} + E_p^{k})\sin \sigma p = 2\cos k\sigma \sin \sigma p$$

$$(E_p^{-k} - E_p^{k}) \cos \sigma p \quad = \quad 2 \sin k\sigma \sin \sigma p$$

$$(E_p^{-k} + E_p^{k}) \cos \sigma p \quad = \quad 2 \cos k\sigma \cos \sigma p$$

$$(E_p^{-k} - E_p^{k}) \,(-1)^p \sinh \sigma p \quad = \quad 2 \sinh k\sigma \,(-1)^p \cosh \sigma p$$

$$(E_p^{-k} + E_p^{k}) \,(-1)^p \sinh \sigma p \quad = -2 \cosh k\sigma \,(-1)^p \sinh \sigma p$$

$$(E_p^{-k} - E_p^{k}) \,(-1)^p \cosh \sigma p \quad = \quad 2 \sinh k\sigma \,(-1)^p \sinh \sigma p$$

$$(E_p^{-k} + E_p^{k}) \,(-1)^p \cosh \sigma p \quad = -2 \cosh k\sigma \,(-1)^p \cosh \sigma p$$

$$E_p^{-k}\, e^{\sigma p} = e^{\sigma(p-k)}$$

$$E_p^{k} \cos \sigma p = \cos(p+k)\sigma$$

APPENDIX B

DEFINITIONS FOR λ AND ϵ

The constant terms appearing in the expressions for $(\psi_r)_{r,s}$ and $(\psi_s)_{r,s}$ (eq. (36) and (37)) are as follows:

$$\lambda_1 = -\sinh\alpha\,\{\,[\,39\cos(p\pi/n) + 96 + \bar{\gamma}\,(116\cos(p\pi/n) + 274)\,]\,[\cosh\alpha\,(17\cos(p\pi/n)$$
$$+ 22) + 28\cos(p\pi/n) + 68 + 10\,\bar{\gamma}(\cosh\alpha + 2)\,(3\cos(p\pi/n) - 4)]$$
$$+ 28\,\bar{\gamma}\sin^2(p\pi/n)\,[39\cosh\alpha + 96 + \bar{\gamma}\,(116\cosh\alpha + 274)]\}\,/\lambda_7$$

$$\lambda_2 = \sin\beta\,\{[39\cos(p\pi/n) + 96 + \bar{\gamma}\,(116\cos(p\pi/n) + 274)\,]\,[\cos\beta\,(17\cos(p\pi/n) + 22)$$
$$+ 28\cos(p\pi/n) + 68 + 10\,\bar{\gamma}\,(\cos\beta + 2)\,(3\cos(p\pi/n) - 4)] + 28\,\bar{\gamma}\sin^2(p\pi/n)$$
$$[39\cos\beta + 96 + \bar{\gamma}(116\cos\beta + 274)\,]\,\}\,/\lambda_8$$

λ_3 is the same as λ_1 with sinh α replaced by $-$ sinh ξ and cosh α by $-$ cosh ξ. The denominator is now λ_9.

$$\lambda_4 = -\sin(p\pi/n)\,\{[39\cosh\alpha + 96 + \bar{\gamma}\,(116\cosh\alpha + 274)]\,[\cos(p\pi/n)\,(17\cosh\alpha + 22)$$
$$+ 28\cosh\alpha + 68 + 10\,\bar{\gamma}\,(\cos(p\pi/n) + 2)\,(3\cosh\alpha - 4)\,]$$
$$- 28\,\bar{\gamma}\sinh^2\alpha\,[39\cos(p\pi/n) + 96 + \bar{\gamma}(116\cos(p\pi/n) + 274)\,]\}\,/\lambda_7$$

$$\lambda_5 = -\sin(p\pi/n)\,\{\,[39\cos\beta + 96 + \bar{\gamma}\,(116\cos\beta + 274)\,]\,[\cos(p\pi/n)\,(17\cos\beta + 22)$$
$$+ 28\cos\beta + 68 + 10\,\bar{\gamma}\,(\cos(p\pi/n) + 2)\,(3\cos\beta - 4)\,]$$
$$+ 28\,\bar{\gamma}\sin^2\beta\,[39\cos(p\pi/n) + 96 + \bar{\gamma}(116\cos(p\pi/n) + 274)\,]\,\}\,/\lambda_8$$

λ_6 is the same as λ_4 with sinh α replaced by $-$ sinh ξ and cosh α by $-$ cosh ξ. The denominator is now λ_9.

$$\lambda_7 = -784\,\bar{\gamma}^2\sinh^2\alpha\sin^2(p\pi/n) - \{\cosh\alpha\,(17\cos(p\pi/n) + 28) + 22\cos(p\pi/n) + 68$$
$$+ 10\,\bar{\gamma}\,(\cos(p\pi/n) + 2)\,(3\cosh\alpha - 4)\}\,\{\cosh\alpha\,(17\cos(p\pi/n) + 22)$$
$$+ 28\cos(p\pi/n) + 68 + 10\,\gamma\,(\cosh\alpha + 2)\,(3\cos(p\pi/n) - 4)\}$$

$$\lambda_8 = 784\,\bar{\gamma}^2\sin^2\beta\sin^2(p\pi/n) - \{\cos\beta\,(17\cos(p\pi/n) + 28) + 22\cos(p\pi/n) + 68$$
$$+ 10\,\bar{\gamma}\,(\cos(p\pi/n) + 2)\,(3\cos\beta - 4)\}\,\{\cos\beta\,(17\cos(p\pi/n) + 22)$$
$$+ 28\cos(p\pi/n) + 68 + 10\,\gamma(\cos\beta + 2)\,(3\cos(p\pi/n) - 4)\}$$

λ_9 is the same as λ_7 with sinh α replaced by $-$ sinh ξ and cosh α by $-$ cosh ξ.

The constant terms appearing in the expressions for $(M_r)_{q,s}$, $(M_s)_{q,s}$ and $(V\cdot a)_{q,s}$ (eq. (40)) are as follows:

$$\epsilon_1 = \lambda_1 [Z_1 \sinh \alpha(q-1) + Z_2 \sinh \alpha q + 10\,\bar{\gamma}\, Z_3\, (3 \sinh \alpha(q-1) - 4 \sinh \alpha q)\,]$$
$$+ 14\,\bar{\gamma}\,\lambda_4\,(2 \cosh \alpha(q-1) + 3 \cosh \alpha q) \sin(p\pi/n)$$
$$+ Z_4 \cosh \alpha(q-1) - Z_5 \cosh \alpha q + \bar{\gamma}\,[Z_6 \cosh \alpha(q-1) + Z_7 \cosh \alpha q\,]$$

$$\epsilon_2 = \lambda_2 [Z_1 \sin \beta(q-1) + Z_2 \sin \beta q + 10\,\bar{\gamma}\, Z_3\, (3 \sin \beta(q-1) - 4 \sin \beta q)\,]$$
$$+ 14\,\bar{\gamma}\,\lambda_5\,(2 \cos \beta(q-1) + 3 \cos \beta q) \sin(p\pi/n)$$
$$+ Z_4 \cos \beta(q-1) - Z_5 \cos \beta q + \bar{\gamma}\,[Z_6 \cos \beta(q-1) + Z_7 \cos \beta q]$$

$$\epsilon_3 = \lambda_3 [Z_1\,(-1)^{q-1} \sinh \xi(q-1) + Z_2\,(-1)^q \sinh \xi q + 10\,\bar{\gamma}\, Z_3\,(3\,(-1)^{q-1} \sinh \xi(q-1)$$
$$- 4\,(-1)^q \sinh \xi q)] + 14\,\bar{\gamma}\,\lambda_6\,(2\,(-1)^{q-1} \cosh \xi(q-1) + 3\,(-1)^q \cosh \xi q) \sin(p\pi/n)$$
$$+ Z_4\,(-1)^{q-1} \cosh \xi(q-1) - Z_5\,(-1)^q \cosh \xi q + \bar{\gamma}\,[Z_6\,(-1)^{q-1} \cosh \xi(q-1)$$
$$+ Z_7\,(-1)^q \cosh \xi q]$$

$$\epsilon_4 = -\,14\,\lambda_1\,\bar{\gamma}\,[\sin(p\pi/n)\,(2 \sinh \alpha(q-1) - 3 \sinh \alpha q)\,]$$
$$+ \lambda_4\,[Z_8 \cosh \alpha(q-1) + Z_9 \cosh \alpha q + 10\,\bar{\gamma}\, Z_{10}\,(\cosh \alpha(q-1) + 2 \cosh \alpha q)\,]$$
$$- \sin(p\pi/n)\,[39 \cosh \alpha(q-1) + 96 \cosh \alpha q - \bar{\gamma}\,(116 \cosh \alpha(q-1) + 274 \cosh \alpha q)\,]$$

$$\epsilon_5 = -\,14\,\lambda_2\,\bar{\gamma}\,[\sin(p\pi/n)\,(2 \sin \beta(q-1) - 3 \sin \beta q)\,]$$
$$+ \lambda_5\,[Z_8 \cos \beta(q-1) + Z_9 \cos \beta q + 10\,\bar{\gamma}\, Z_{10}\,(\cos \beta(q-1) + 2 \cos \beta q)\,]$$
$$- \sin(p\pi/n)\,[39 \cos \beta(q-1) + 96 \cos \beta q - \bar{\gamma}\,(116 \cos \beta(q-1) + 274 \cos \beta q)\,]$$

$$\epsilon_6 = -\,14\lambda_3\,\bar{\gamma}\,[\sin(p\pi/n)\,(2\,(-1)^{q-1} \sinh \xi(q-1) - 3\,(-1)^q \sinh \xi q)\,]$$
$$+ \lambda_6\,[Z_8\,(-1)^{q-1} \cosh \xi(q-1) + Z_9\,(-1)^q \cosh \xi q + 10\bar{\gamma}\, Z_{10}\,(\,(-1)^{q-1} \cosh \xi(q-1)$$
$$+ 2\,(-1)^q \cosh \xi q)\,] - \sin(p\pi/n)\,[39\,(-1)^{q-1} \cosh \xi(q-1) + 96\,(-1)^q \cosh \xi q$$
$$- \bar{\gamma}\,(116\,(-1)^{q-1} \cosh \xi(q-1) + 274\,(-1)^q \cosh \xi q)\,]$$

$$\epsilon_7 = -\,\lambda_1\,[Z_4 \sinh \alpha(q-1) + Z_5 \sinh \alpha q + \bar{\gamma}\,(Z_6 \sinh \alpha(q-1) - Z_7 \sinh \alpha q)\,]$$
$$- \lambda_4 \sin(p\pi/n)\,[39 \cosh \alpha(q-1) + 96 \cosh \alpha q + \bar{\gamma}\,(116 \cosh \alpha(q-1) + 274 \cosh \alpha q)\,]$$
$$- 3\,Z_2 \cosh \alpha(q-1) - Z_{11} \cosh \alpha q - \bar{\gamma}\,(Z_{12} \cosh \alpha(q-1) + Z_{13} \cosh \alpha q)$$

$$\epsilon_8 = -\lambda_2 \,[Z_4 \sin\beta(q-1) + Z_5 \sin\beta q + \bar{\gamma}\,(Z_6 \sin\beta(q-1) - Z_7 \sin\beta q)\,]$$

$$- \lambda_5 \sin(p\pi/n)\,[39 \cos\beta(q-1) + 96 \cos\beta q + \bar{\gamma}\,(116 \cos\beta(q-1) + 274 \cos\beta q)\,]$$

$$- 3\, Z_2 \cos\beta(q-1) - Z_{11} \cos\beta q - \bar{\gamma}\,(Z_{12} \cos\beta(q-1) + Z_{13} \cos\beta q)$$

$$\epsilon_9 = -\lambda_3 \,[Z_4\,(-1)^{q-1} \sinh\xi(q-1) + Z_5\,(-1)^q \sinh\xi q + \bar{\gamma}\,(Z_6\,(-1)^{q-1} \sinh\xi(q-1)$$

$$- Z_7\,(-1)^q \sinh\xi q)\,] - \lambda_6 \sin(p\pi/n)\,[39\,(-1)^{q-1} \cosh\xi(q-1)$$

$$+ 96\,(-1)^q \cosh\xi q + \bar{\gamma}\,(116\,(-1)^{q-1} \cosh\xi(q-1) + 274\,(-1)^q \cosh\xi q)\,]$$

$$- 3\, Z_2\,(-1)^{q-1} \cosh\xi(q-1) - Z_{11}\,(-1)^q \cosh\xi q - \bar{\gamma}\,[Z_{12}\,(-1)^{q-1} \cosh\xi(q-1)$$

$$+ Z_{13}\,(-1)^q \cosh\xi q]$$

where

$Z_1 = 17\cos(p\pi/n) + 28$; $Z_2 = 22\cos(p\pi/n) + 68$; $Z_3 = \cos(p\pi/n) + 2$;

$Z_4 = 39\cos(p\pi/n) + 96$; $Z_5 = 24\cos(p\pi/n) + 111$; $Z_6 = 116\cos(p\pi/n) + 274$;

$Z_7 = 199\cos(p\pi/n) + 461$; $Z_8 = 17\cos(p\pi/n) + 22$; $Z_9 = 28\cos(p\pi/n) + 68$;

$Z_{10} = 3\cos(p\pi/n) - 4$; $Z_{11} = 204\cos(p\pi/n) - 474$;

$Z_{12} = 394\cos(p\pi/n) + 1226$; $Z_{13} = 1226\cos(p\pi/n) + 3454$